Kirchhoff modeling for attenuative anisotropic media

Bharath Shekar & Ilya Tsvankin
Center for Wave Phenomena, Colorado School of Mines, Golden CO 80401

ABSTRACT
Seismic wave propagation in attenuative media can be efficiently modeled with ray-based methods. Here, we present a methodology to generate reflection data from attenuative anisotropic media using the Kirchhoff scattering integral and summation of Gaussian beams. Green's functions are computed in the reference elastic model by Gaussian-beam summation, and the influence of attenuation is incorporated as a perturbation along the central ray. The reflected wavefield is obtained by substituting the approximate Green's functions into the Kirchhoff scattering integral. Numerical examples for P-waves in transversely isotropic (TI) media with a horizontal reflector demonstrate the accuracy of the method.

Key words: attenuation, anisotropy, Kirchhoff modeling, Gaussian beams

Introduction

Attenuation analysis can provide seismic attributes sensitive to the physical properties of the subsurface. Reliable attenuation measurements have become feasible with acquisition of high-quality reflection and borehole data.

A prerequisite for estimating attenuation coefficients from seismic data is accurate and efficient modeling of wave propagation in attenuative media. The stiffness tensor in attenuative media is complex, which leads to amplitude decay along seismic rays and velocity dispersion. In the presence of attenuation, stress is obtained by convolving the time domain stiffness tensor (called the relaxation tensor) with the strain tensor (Carcione, 1990), which complicates time-domain finite-difference modeling of wave propagation. Further, to simulate a frequency-independent quality factor ("constant-Q" model, e.g., Kjartannson, 1979) it is essential to superimpose various relaxation mechanisms (Xu and McMechan, 1998; Ruud and Hestholm, 2005), thus increasing the cost of finite-difference modeling. The approach based on the Fourier pseudospectral method proposed by Carcione (2011) avoids the computation of relaxation functions, but it is restricted to viscoacoustic media. The reflectivity method (e.g., Schmidt and Tango, 1986) is limited to calculating exact synthetic seismograms for laterally homogeneous (elastic or attenuative) models with plane interfaces.

A computationally efficient alternative is ray tracing, which can generate asymptotic Green's functions in both elastic and attenuative models (Červený, 2001). So-called "complex" ray theory developed for attenuative models treats ray trajectories and parameters computed along the ray as complex quantities (Thomson, 1997; Hanyga and Seredyňska, 2000). However, numerical implementation of "complex" ray theory in seismic modeling is not straightforward. Ray tracing in the presence of attenuation can also be performed using perturbation methods, which involve computation of rays in a reference elastic medium with the influence of attenuation included as a perturbation along the ray (Gajewski and Pšenčik, 1992; Červený and Pšenčik, 2009; Shekar and Tsvankin, 2012).

Synthetic seismograms of reflected waves in heterogeneous media can be computed using the Kirchhoff scattering integral (Chapman, 2004). However, this method typically requires two-point ray tracing, which does not properly describe multivalued traveltimes (multipathing). Alternatively, the asymptotic Green's functions required in the Kirchhoff scattering integral can be generated by summation of Gaussian beams (Bleistein, 2008; Červený, 2001). Gaussian-beam summation can accurately handle multipathing and produce finite-frequency sensitivity kernels for amplitude inversion (Yomogida and Aki, 1987).

Here, we present an algorithm for computing 2.5D ray synthetic seismograms from attenuative anisotropic media. First, we describe the Kirchhoff scattering integral for purely elastic models and show how it should be modified in the presence of attenuation. Then we review the method of summation of Gaussian beams and its application to computation of the asymptotic "two-point" Green's functions in attenuative media. Finally, this methodology is implemented for TI media and its accuracy is illustrated with numerical examples.

Methodology

0.1 Kirchhoff scattering integral

For simplicity, here we describe the computation of synthetic seismograms for a single reflecting interface. Our treatment is restricted to P-waves and does not include P-to-S mode conversions. The sources and receivers are assumed to be distributed on the same surface, which lies above the reflector. Suppose the wavefield is excited by a point force located at $\mathbf{x}_s$ and aligned with the x_k-axis, and the receiver is located at $\mathbf{x}_r$. The nth component of the displacement field in the frequency domain is given by (Červený, 2001):

$$G_{nk}(\mathbf{x}_r, \mathbf{x}_s, \omega) = -\,i\,\omega \int_{\Sigma} \mathcal{W}_{iq}(\mathbf{x}') \times G_{in}(\mathbf{x}', \mathbf{x}_s, \omega)\, G_{qk}(\mathbf{x}', \mathbf{x}_r, \omega)\, \mathrm{d}\Sigma\,, \tag{1}$$

where $\mathbf{x}'$ are points on the scattering surface Σ, and the source- and receiver-side Green's functions, $G_{in}(\mathbf{x}', \mathbf{x}_s, \omega)$ and $G_{qk}(\mathbf{x}', \mathbf{x}_r, \omega)$, are computed for a smoothed medium. The weighting function $\mathcal{W}_{iq}(\mathbf{x}')$ is represented as:

$$\mathcal{W}_{iq}(\mathbf{x}') = a^{(1)}_{ijql}\,(n_j\, p^r_l - n_l\, p^s_j)\,(1+R)\,, \tag{2}$$

where $a^{(1)}_{ijql}$ is the local density-normalized stiffness tensor in the medium above the reflector, n_j is the normal to the reflector, $\mathbf{p}^s$ and $\mathbf{p}^r$ are the source- and receiver-side slowness vectors (respectively) at the scattering point, and R is the PP-wave reflection coefficient.

Equation 1 is valid for an arbitrary scattering surface, and all Green's functions have to be computed in 3D. However, if the medium properties do not vary in the x_2-direction, and the plane $[x_1, x_3]$ is a plane of symmetry, equation 1 can be represented in a 2.5D form. Then the surface integral in equation 1 can be reduced to a line integral by the method of stationary phase (Bleistein, 1984).

Here, we use the results of Bleistein (1986), who evaluates an integral similar to that in equation 1 by the stationary-phase method. The 2.5D version of equation 1 can be obtained as:

$$G_{nk}(\mathbf{x}_r, \mathbf{x}_s, \omega) = -\,i\,\sqrt{2\,\pi\,\omega} \int_{C,\, x_2=0} \frac{1}{\sigma}\, \mathcal{W}_{iq}(\mathbf{x}') \times G_{in}(\mathbf{x}', \mathbf{x}_s, \omega)\, G_{qk}(\mathbf{x}', \mathbf{x}_r, \omega)\, \mathrm{d}s, \tag{3}$$

where the Green's functions are defined in 2.5D, the scatterer is reduced to the curve C that lies in the $[x_1, x_3]$-plane, ds is an elementary arc-length along C, and function σ accounts for out-of-plane phenomena:

$$\sigma = \Big[\frac{\partial^2 T(\mathbf{x}', \mathbf{x}_s)}{\partial x_2^2} + \frac{\partial^2 T(\mathbf{x}', \mathbf{x}_r)}{\partial x_2^2}\Big]_{x_2=0}\,; \tag{4}$$

$T(\mathbf{x}', \mathbf{x}_s)$ is the traveltime from the scatterer to the source and $T(\mathbf{x}', \mathbf{x}_r)$ is the traveltime from the scatterer to the receiver. The second-order spatial derivatives of the traveltime functions may be calculated from dynamic ray tracing.

Equations 1-4 can be extended to attenuative media by making the stiffness tensor complex and replacing the elastic Green's functions with their viscoelastic counterparts. Although the reflection coefficient and slowness vector also become complex in attenuative media, we compute these quantities for the reference elastic medium. With the exception of anomalously high attenuation, plane-wave reflection coefficients are not significantly distorted by attenuation (Behura and Tsvankin, 2009). While the complex-valued slowness vectors at the reflector can somewhat alter the weighting function defined in equation 2, they do not significantly contribute to the displacement computed from equation 3 because attenuation is mostly a propagation phenomenon.

0.2 Asymptotic Green's function as a sum of Gaussian beams

Although the Green's functions in equation 3 can be computed by two-point ray tracing (Bulant, 1996), that method cannot accurately handle multipathing. A more rigorous approach to modeling asymptotic Green's functions involves summation of Gaussian beams (Červený, 2001). Here, we start with the computation of 2.5D elastic Green's functions and then describe the modifications needed for extending the methodology to attenuative media.

The Green's function $\mathbf{G}(\mathbf{x}', \mathbf{x}_s, \omega)$ can be found as the following sum of Gaussian beams (Červený, 2001):

$$\mathbf{G}(\mathbf{x}', \mathbf{x}_s, \omega) = \int \Phi(\gamma_1)\, \mathbf{u}_{GB}(R_0(\gamma_1))\, \mathrm{d}\gamma_1\,, \tag{5}$$

where $\mathbf{u}_{GB}(R_0(\gamma_1))$ represents a single Gaussian beam concentrated around a central ray $R_0(\gamma_1)$, $\Phi(\gamma_1)$ is a weighting function, and γ_1 is an initial value of a certain ray parameter (e.g., of the phase angle). Suppose that the ray $R_0(\gamma_1^{\,0})$ illuminates a point close to $\mathbf{x}'$. The range of integration is then chosen to be symmetric over $\gamma_1^{\,0}$, and the Green's function is obtained by summation over a fan of beams that originate at the source location $\mathbf{x}_s$ and illuminate a region around $\mathbf{x}'$.

To evaluate the contribution of $\mathbf{u}_{GB}(R_0(\gamma_1))$ to $\mathbf{G}(\mathbf{x}', \mathbf{x}_s, \omega)$, we consider the point $\mathbf{x}''$ closest to $\mathbf{x}'$ on the central ray R_0. Then the contribution of $\mathbf{u}_{GB}(R_0(\gamma_1))$ to $\mathbf{G}(\mathbf{x}', \mathbf{x}_s, \omega)$ is (Červený and Pšenčik, 2010):

$$\mathbf{u}_{GB}(\mathbf{x}'', \mathbf{x}_s, \omega) = \frac{c(\mathbf{x}_s)}{c(\mathbf{x}'')}\, \mathbf{g}(\mathbf{x}'') \times \frac{1}{\sqrt{\det \tilde{\mathbf{W}}(\mathbf{x}'', \mathbf{x}_s)}}\, e^{-i\,\omega\, \tilde{T}(\mathbf{x}', \mathbf{x}_s)}\,, \tag{6}$$

where c is the phase velocity, $\mathbf{g}$ is the polarization vector, and $\tilde{T}(\mathbf{x}', \mathbf{x}_s)$ is the complex traveltime:

$$\tilde{T}(\mathbf{x}', \mathbf{x}_s) = T(\mathbf{x}'', \mathbf{x}_s) + (\mathbf{x}' - \mathbf{x}'')^T \, \mathbf{p}(\mathbf{x}'') + \frac{1}{2} (\mathbf{x}' - \mathbf{x}'')^T \, \tilde{\mathbf{M}}^x \, (\mathbf{x}' - \mathbf{x}'') \,, \qquad (7)$$

where the superscripts "T" denotes the transpose, $\mathbf{p}$ is the slowness vector, and $\tilde{\mathbf{M}}^x$ is the complex-valued matrix of the second derivatives of the traveltime in the Cartesian coordinates. The matrix $\tilde{\mathbf{M}}^x$ is found by transforming the matrix $\tilde{\mathbf{M}}$ defined in Appendix A (equation 11) to the Cartesian coordinates (Červený and Pšenčik, 2010). The matrix $\tilde{\mathbf{W}}(\mathbf{x}'', \mathbf{x}_s)$ depends on the initial value $\tilde{\mathbf{M}}_0$ of the complex-valued matrix $\tilde{\mathbf{M}}$:

$$\tilde{\mathbf{W}}(\mathbf{x}'', \mathbf{x}_s) = \mathbf{Q}_1(\mathbf{x}'', \mathbf{x}_s) + \mathbf{Q}_2(\mathbf{x}'', \mathbf{x}_s) \, \tilde{\mathbf{M}}_0 \,, \qquad (8)$$

where the matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are computed by dynamic ray tracing in the ray-centered coordinates (see Appendix A). The matrix $\tilde{\mathbf{M}}_0$ is given by:

$$\tilde{\mathbf{M}}_0 = \frac{i}{l \, \omega^2} \, \mathbf{I} \,, \qquad (9)$$

where $\mathbf{I}$ is the identity matrix, and l represents the beam width. In anisotropic media, the beam width can be chosen as (Alkhalifah, 1995):

$$l = \frac{V_{\text{avg}}}{f_{\text{min}}} \,, \qquad (10)$$

where V_{avg} represents the average of the horizontal and vertical phase velocities over the entire model, and f_{min} is the minimum frequency of the source signal.

The weighting function $\Phi(\gamma_1)$ (equation 5) can be found from the matrices $\mathbf{Q}_2$, $\mathbf{M}$, and $\tilde{\mathbf{M}}$ (Červený, 2001):

$$\Phi(\gamma_1) = \sqrt{\frac{\omega}{2\,\pi}} \left| \det \mathbf{Q}_2(\mathbf{x}'', \mathbf{x}_s) \right| \times \sqrt{\det \left[\mathbf{M}(\mathbf{x}'', \mathbf{x}_s) - \tilde{\mathbf{M}}(\mathbf{x}'', \mathbf{x}_s) \right]} \,. \qquad (11)$$

The real-valued matrices $\mathbf{Q}_2$ and $\mathbf{M}$ are computed by dynamic ray tracing with point-source initial conditions (Appendix A), and the complex-valued matrix $\tilde{\mathbf{M}}$ is computed from equation 11, Appendix A.

In attenuative media, equation 5 can be modified to obtain the viscoelastic Green's function $\mathbf{G}^{\text{att}}(\mathbf{x}', \mathbf{x}_s, \omega)$ (Červený, 1985):

$$\mathbf{G}^{\text{att}}(\mathbf{x}', \mathbf{x}_s, \omega) = \int \Phi(\gamma_1) \, \mathbf{u}_{GB}^{\text{att}}(R_0(\gamma_1)) \, \mathrm{d}\gamma_1 \,. \qquad (12)$$

The weighting function $\Phi(\gamma_1)$ remains unchanged (equation 11), whereas the Gaussian-beam displacement $\mathbf{u}_{GB}^{\text{att}}(R_0)$ becomes

$$\mathbf{u}_{GB}^{\text{att}}(R_0(\gamma_1)) = \mathbf{u}_{GB}(R_0(\gamma_1)) \, e^{-\omega \, t^*(\mathbf{x}', \mathbf{x}_s)} \,, \qquad (13)$$

where $\mathbf{u}_{GB}(R_0(\gamma_1))$ is the Gaussian beam computed for the reference elastic medium. The real-valued quantity $t^*(\mathbf{x}', \mathbf{x}_s)$, called the "dissipation factor" (Gajewski and Pšenčik, 1992), accounts for the attenuation-induced amplitude decay along the central ray. The dissipation factor can be calculated using perturbation methods (Červený and Pšenčik, 2009; Shekar and Tsvankin, 2012).

0.3 Implementation

The reflected wavefields in attenuative heterogeneous media are calculated using equation 3 with the source-to-scatterer and scatterer-to-receiver Green's functions obtained from equation 6. The Gaussian beams (equation 7) are computed in the reference elastic medium with the dissipation factor found as a perturbation along the central ray (Červený and Pšenčik, 2009; Shekar and Tsvankin, 2012). The weighting function $\Phi(\gamma_1)$ for the summation of Gaussian beams is also calculated in the reference elastic medium. Likewise, the weighting functions σ and $\mathcal{W}_{iq}$ for the Kirchhoff integral (equation 3) are found from the quantities stored during the modeling of Gaussian beams in the elastic background. For TI models, the reflection coefficient R in equation 2 is obtained from the weak-contrast, weak-anisotropy approximation presented by Rüger (1997).

The outlined method involves a number of approximations. The Kirchhoff scattering integral itself is an asymptotic solution that ignores multiple scattering (Chapman, 2004). The method of summation of Gaussian beams is limited to computing asymptotic Green's functions in smooth media. The influence of attenuation is modeled using perturbation theory, which is valid for weakly-to-moderately dissipative media. Numerical examples illustrating the accuracy of the perturbation approach can be found in Shekar and Tsvankin (2012).

Numerical examples

First, we verify the accuracy of the Gaussian beam summation method in constructing the asymptotic Green's function. Figure 1 compares the displacements computed from perturbation ray theory (U_{ART}) and Gaussian beam summation (U_{GB}) for a homogeneous VTI model. Even though the medium is strongly dissipative with the P- and S-wave vertical quality factors equal to 10, perturbation theory is sufficiently accurate (Shekar and Tsvankin, 2012), and the two displacements are close to one another.

Next, we test the accuracy of the Kirchhoff scattering integral combined with the Gaussian beam summation method in generating reflection data. Table 1 displays the velocity and attenuation parameters for a laterally homogeneous VTI medium above a horizontal reflector. The exact reflected wavefield (Figure 2a) was generated using the reflectivity method (Mallick and Frazer, 1990). The data computed by the Kirchhoff scattering integral (Figure 2b) have a similar amplitude but

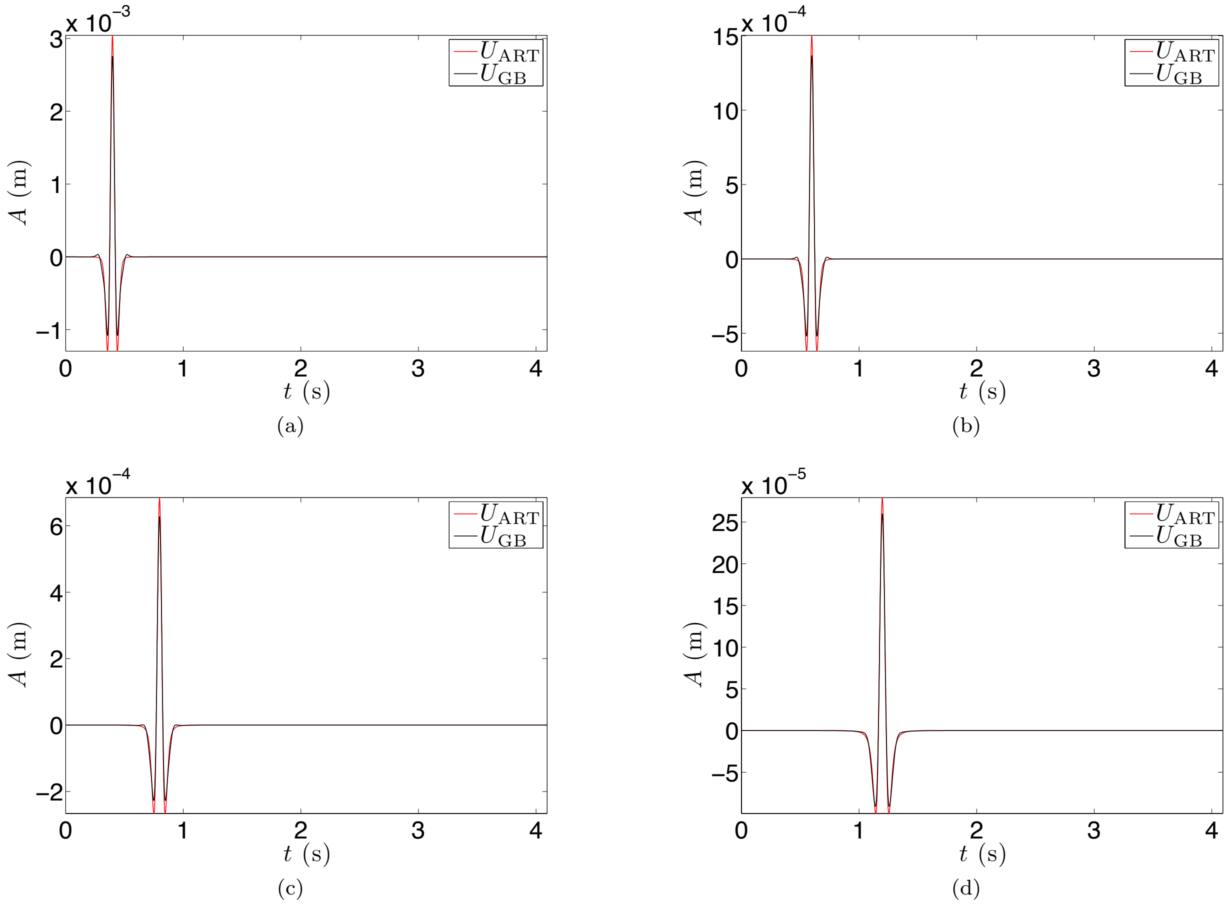

Figure 1. Comparison of the vertical displacement component computed using perturbation ray theory (U_{ART}, red line) and Gaussian-beam summation method (U_{GB}, black line) for group angles with the vertical of (a) 0°, (b) 30°, (c) 60°, and (d) 90°. The model is homogeneous VTI with the P-wave vertical velocity $V_{P0} = 2.0$ km/s, S-wave vertical velocity $V_{S0} = 1.0$ km/s, and anisotropy parameters $\epsilon = 0.4$ and $\delta = 0.25$. The P-wave vertical quality factor is $Q_{P0} = 10$, S-wave vertical quality factor $Q_{S0} = 10$, and the attenuation-anisotropy parameters (defined in Zhu and Tsvankin, 2006) are $\epsilon_Q = -0.45$ and $\delta_Q = -0.50$. The wavefield is excited by a vertical point force; the source signal is a Ricker wavelet with a central frequency of 10 Hz.

lower frequency content than the seismograms in Figure 2a because the Gaussian beam summation method yields finite-frequency Green's functions. The Kirchhoff scattering integral also produces spurious events near both ends of the receiver array, so these traces have been muted out.

Conclusions

We introduced a ray-based methodology for computing synthetic seismograms of reflected waves from attenuative anisotropic media. The wavefield is generated with the Kirchhoff scattering integral that includes 2.5D asymptotic Green's functions computed using Gaussian beam summation and perturbation theory.

The accuracy of the Gaussian-beam summation method in producing Green's functions was verified for highly attenuative TI media. We also compared the Kirchhoff scattering integral with the exact seismograms computed using the reflectivity method. The examples confirm that the proposed technique adequately models P-wave reflections even in the presence of strong anisotropic attenuation. However, the frequency content of the data produced by the Kirchhoff scattering integral is lower than that of the exact seismograms.

Acknowledgments

We are grateful to the members of the A(nisotropy)-Team of the Center for Wave Phenomena (CWP), Colorado School of Mines, for fruitful discussions. Support for this work was provided by the Consortium Project on Seismic Inverse Methods for Complex Structures at CWP.

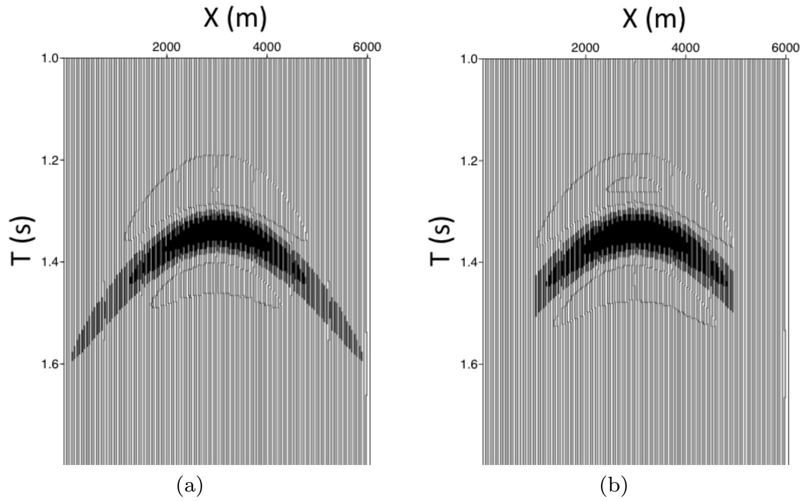

Figure 2. Vertical displacement for the model in Table 1 generated using (a) the reflectivity method and (b) the Kirchhoff scattering integral. The wavefield is excited and recorded on top of the model. The source is a vertical force at $X = 3.0$ km and the receivers are placed between $X = 0.5$ km and $X = 5.5$ km with a 50 m increment. The source signal is a Ricker wavelet with a central frequency of 10 Hz.

	Top layer	Bottom half-space
Thickness (km)	2.00	–
V_{P0} (km/s)	3.00	3.20
V_{S0} (km/s)	1.50	1.60
ϵ	0.20	0.10
δ	0.10	0.05
Q_{P0}	10	10
Q_{S0}	10	10
ϵ_Q	0.0	0.0
δ_Q	0.0	0.0

Table 1. Synthetic VTI model used to test the Kirchhoff scattering integral. The P-wave is reflected from a horizontal interface between the two media.

REFERENCES

Alkhalifah, T., 1995, Gaussian beam depth migration for anisotropic media: Geophysics, **60**, 1474–1484.

Bleistein, N., 1984, Mathematical Methods for Wave Phenomena: Academic Press.

———, 1986, Two-and-one-half dimensional in-plane wave propagation: Geophysical Prospecting, **34**, 686–703.

———, 2008, Seismic wavefields in layered isotropic media (course notes): CWP, Colorado School of Mines (http://www.cwp.mines.edu/norm/ShrtCrse).

Bulant, P., 1996, Amplitude and phase data inversion for phase velocity anomalies in the Pacific Ocean basin: Pure and Applied Geophysics, **148**, 421–447.

Carcione, J., 1990, Wave propagation in anisotropic linear viscoelastic media: theory and simulated wavefields: Geophysics Journal International, **101**, 739–750.

———, 2011, A generalization of the Fourier pseudospectral method: Geophysics, **76**.

Červený, V., 1985, Gaussian beam synthetic seismograms: Journal of Geophysics, **58**, 44–72.

———, 2001, Seismic ray theory: Cambridge University Press.

Červený, V., and L. Klimeš, 2010, Transformation relations for second-derivatives of traveltime in anisotropic media: Studia Geophysica et Geodaetica, **54**, 257–267.

Červený, V., and I. Pšenčik, 2009, Perturbation Hamiltonians in heterogeneous anisotropic weakly dissipative media: Geophysics Journal International, **178**, 939–949.

———, 2010, Gaussian beams in inhomogeneous anisotropic layered structures: Geophysics Journal International, **180**, 798–812.

Chapman, C., 2004, Fundamentals of Seismic Wave Propagation: Cambridge University Press.

Gajewski, D., and I. Pšenčik, 1992, Vector wavefield for weakly attenuating anisotropic media by the ray method: Geophysics, **57**, 27–38.

Hanyga, A., and M. Seredyňska, 2000, Ray tracing in elastic and viscoelastic media: Pure and Applied Geophysics, **157**, 679–717.

Kjartannson, E., 1979, Constant Q-Wave Propagation and Attenuation: Journal of Geophysical Research,

84, 4737–4748.

Mallick, S., and N. L. Frazer, 1990, Computation of Synthetic Seismograms for Stratified Azimuthally Anisotropic Media: Journal of Geophysical Research, **95**, 8513–8526.

Rüger, A., 1997, P-wave reflection coefficients for transversely isotropic models with vertical and horizontal axis of symmetry: Geophysics, **62**, 713–722.

Ruud, B. O., and S. Hestholm, 2005, Modeling seismic waves in orthorhombic, viscoelastic media by finite-differences: 75th Annual International Meeting, SEG, Expanded Abstracts.

Schmidt, H., and G. Tango, 1986, Efficient global matrix approach to the computation of synthetic seismograms: Geophysical Journal of the Royal Astronomical Society, **84**, 331–359.

Shekar, B., and I. Tsvankin, 2012, Attenuation analysis for heterogeneous transversely isotropic media: 82nd Annual International Meeting, SEG, Expanded Abstracts.

Thomson, C. J., 1997, Complex rays and wave packets for decaying signals in inhomogeneous, anisotropic and anelastic media: Studia Geophysica et Geodaetica, **41**, 345–381.

Xu, T., and G. A. McMechan, 1998, Efficient 3-D viscoelastic modeling with application to near-surface land seismic data: Geophysics, **63**, 601–612.

Yomogida, K., and K. Aki, 1987, Amplitude and phase data inversion for phase velocity anomalies in the Pacific Ocean basin: Geophysical Journal of the Royal Astronomical Society, **88**, 161–204.

Zhu, Y., and I. Tsvankin, 2006, Plane-wave propagation in attenuative transversely isotropic media: Geophysics, **71**, no.2, T17–T30.

Appendix A

Dynamic ray tracing and Gaussian beams in anisotropic media

In this section, we briefly review dynamic ray tracing in anisotropic media and introduce the quantities necessary for the construction of Gaussian beams.

The eikonal equation in elastic, anisotropic, heterogeneous media can be written as (Červený, 2001):

$$G(x_i, p_i) = 1\,, \tag{1}$$

where x_i are the spatial coordinates and p_i are the components of the slowness vector. The solutions of equation 1 represent the eigenvalues of the Christoffel equation:

$$\det\left[\Gamma_{ik} - G\,\delta_{ik}\right] = 0\,, \tag{2}$$

where $\Gamma_{ik} = a_{ijkl}\,p_j\,p_l$ are the components of the Christoffel matrix and a_{ijkl} form the density-normalized stiffness tensor.

The kinematic ray-tracing equations are given by (Červený, 2001):

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i}\,, \tag{3}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i}\,, \tag{4}$$

where τ represents the traveltime (eikonal) along the ray.

The dynamic ray tracing system in ray-centered coordinates can be represented as (Červený and Klimeš, 2010):

$$\begin{aligned}\frac{\mathrm{d}Q_{NI}}{\mathrm{d}\tau} &= A_{NM}\,Q_{MI} + B_{NM}\,P_{MI}\,,\\ \frac{\mathrm{d}P_{NI}}{\mathrm{d}\tau} &= -C_{NM}\,Q_{MI} - D_{NM}\,P_{MI}\,,\end{aligned} \tag{5}$$

where the indices N, M, and I vary from 1 to 2. Explicit expressions for matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ can be found in Červený and Klimeš (2010). The matrices Q_{NI} and P_{NI} are defined as

$$Q_{NI} = \frac{\partial q_N}{\partial \gamma_I}\,,\, P_{NI} = \frac{\partial p_N}{\partial \gamma_I}\,,$$

where γ_I is a certain "ray parameter" (e.g., the initial phase angle of the ray), q_N are the coordinates tangent to the wavefront, and p_N denotes the slowness vector in the ray-centered coordinate system:

$$p_N = \frac{\partial \tau}{\partial q_N}\,. \tag{6}$$

The solution of system 5 for plane-wave initial conditions ($\mathbf{Q} = \mathbf{I}$, $\mathbf{P} = 0$; $\mathbf{I}$ is the identity matrix) is denoted by $\mathbf{Q}_1$ and $\mathbf{P}_1$, and for point-source initial conditions ($\mathbf{Q} = 0$, $\mathbf{P} = \mathbf{I}$) by $\mathbf{Q}_2$ and $\mathbf{P}_2$.

It is convenient to introduce the real-valued matrix $\mathbf{M}$ of the second-order traveltime derivatives:

$$\mathbf{M} = \mathbf{P}\,\mathbf{Q}^{-1}\,. \tag{7}$$

as discussed in the main text (equation 7), the matrix $\mathbf{M}$ is used for computing the paraxial traveltime. For point-source initial conditions, $\mathbf{M}$ is

$$\mathbf{M} = \mathbf{P}_2\,\mathbf{Q}_2^{-1}\,. \tag{8}$$

A Gaussian beam is a solution of system 5 with complex-valued initial conditions (Bleistein, 2008):

$$\mathbf{Q} = \frac{l\,\omega^2}{c_0}\,\mathbf{I}\,,\;\; \tilde{\mathbf{P}} = \frac{i}{c_0}\,\mathbf{I}\,, \tag{9}$$

where l is the initial value of the beam width, ω is the angular frequency, and c_0 is the phase velocity at the take-off point. The matrix $\mathbf{M}$ becomes complex-valued:

$$\tilde{\mathbf{M}} = [\mathbf{P}_1 + \tilde{\mathbf{M}}_0\,\mathbf{P}_2][\mathbf{Q}_1 + \tilde{\mathbf{M}}_0\,\mathbf{Q}_2]^{-1}\,, \tag{10}$$

where $\mathbf{P}_1$, $\mathbf{Q}_1$, $\mathbf{P}_2$, and $\mathbf{Q}_2$ are found by dynamic ray tracing. Since $\tilde{\mathbf{M}}$ is complex-valued, the traveltime is complex-valued, which leads to amplitude decay in the direction perpendicular to the central ray. The initial value of $\tilde{\mathbf{M}}$ is

$$\tilde{\mathbf{M}}_0 = \frac{i}{l\,\omega^2}\,\mathbf{I}\,. \tag{11}$$