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Synthetic aperture Green's function retrieval

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ABSTRACT
The cross correlation of noise at two receivers has been used extensively to determine the waves propagating between receivers. This method presumes that noise is incident with equal energy from all directions. In applications when noise is incident from one direction, this leads, in general, to biased estimates of the waves propagating between these receives. We present a method to obtain the true wave velocity from noise propagating in one unknown direction by using three receivers that are not located on a line. Raising the normalized cross correlations of noise recorded at pairs of receivers to appropriately chosen powers is equivalent to projecting the relative location of receivers onto all azimuths over a full circle. By averaging over these azimuths one can retrieve the wave velocity between the receivers and the direction of the propagating noise.

1 INTRODUCTION

The extraction of the earth response from cross correlation of recorded noise has rapidly developed over the last decade. The literature on this topic is extensive, and has been summarized in review papers (Larose et al., 2006; Curtis et al., 2006; Wapenaar et al., 2010a,b; Ritzwoller et al., 2011; Snieder and Larose, 2013), a special issue (Campillo et al., 2011), and books (Wapenaar et al., 2008; Schuster, 2009). The principle that the system response, or Green's function, can be extracted from noise is based on the concept of equipartitioning which states that the noise is either equally distributed over all modes of the system (Lobkis and Weaver, 2001), or that it propagates with equal strength in all directions of propagation (Weaver and Lobkis, 2005). In practice the noise is excited by localized sources, such as storms at the oceans (Stehly et al., 2006), industrial sources (Miyazawa et al., 2008), a concrete dam (O'Connell, 2007), or other sources (Mulargia and Castellaro, 2008; Mulargia, 2012), which violates equipartitioning.

To illustrate the impact of strongly directional noise on Green's function retrieval, we consider the situation where the noise consists of one plane wave with complex spectrum $S(\omega)$ that propagates in the direction of a unit vector $\hat{\mathbf{n}}$ with wave number $k = \omega/c$:

$$u(\mathbf{r}, \omega) = S(\omega) e^{ik\hat{\mathbf{n}}\cdot\mathbf{r}} \ , \tag{1}$$

where ω is the angular frequency and c the wave velocity. The cross correlation $\tilde{C}_{12}(\omega)$ of the noise recorded by receivers at locations $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by

$$\tilde{C}_{12}(\omega) = \langle u(\mathbf{r}_1, \omega) u^*(\mathbf{r}_2, \omega)\rangle = \langle |S(\omega)|^2 \rangle e^{ik\hat{\mathbf{n}}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \ , \tag{2}$$

where the asterisk denotes complex conjugation, and $\langle \cdots \rangle$ the expectation value. This corresponds to a wave that arrives at time $t_{estimated} = \hat{\mathbf{n}} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/c$ instead of the true arrival time $t_{true} = |\mathbf{r}_1 - \mathbf{r}_2|/c$. Since $\hat{\mathbf{n}} \cdot (\mathbf{r}_1 - \mathbf{r}_2) < |\mathbf{r}_1 - \mathbf{r}_2|$ the estimated arrival time is too short, and as a result the estimated velocity is too high. If we knew the direction of wave propagation $\hat{\mathbf{n}}$ we could correct for the bias, but in general this direction is not known.

Several solutions have been proposed to correct for the directional characteristics of noise. The first approach consists of directional filtering of the noise with an array to select the noise that propagates in the desired direction (Curtis and Halliday, 2010). Another approach consists of extracting the coda waves first from cross correlation, and to correlate these coda waves again (Stehly et al., 2008). In this work we present an alternative procedure to extract the correct waves propagating between two receivers when the waves come in from one unknown direction. This procedure is built on the idea that the relative locations between three stations can be projected onto an arbitrary direction. By performing this operation for each direction and integrating over all directions, one can retrieve the wavenumber and direction of propagation of the wave field. We present the theory in section 2 and show a numerical simulation in section 3.

2 THEORY

Consider the geometry of figure 1 with stations at locations $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. We denote the relative locations of

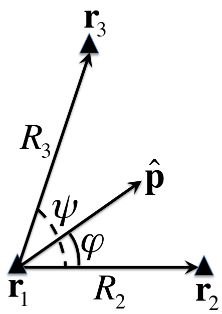

Figure 1. The location of the stations at $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, the definition of the interstation vectors $\mathbf{R}_2$ and $\mathbf{R}_3$, the angle ψ between these vectors, and the polarization vector $\hat{\mathbf{p}}$ at azimuth φ.

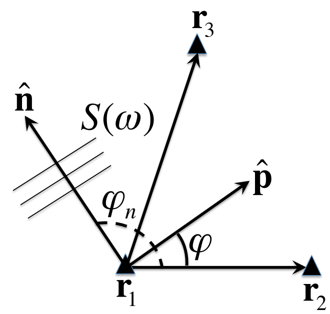

Figure 2. Geometry of figure 1 and the propagation direction $\hat{\mathbf{n}}$ of the propagating noise at azimuth φ_n.

stations 2 and 3 with respect to station 1 by $\mathbf{R}_2 = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{R}_3 = \mathbf{r}_3 - \mathbf{r}_1$, respectively. We denote the angle between these vectors by ψ, see figure 1. The normalized cross correlation of the noise recorded at receivers 2 and 3 with the recorded noise at receiver 1 is defined as $C_{12}(\omega) = \tilde{C}_{12}/|\tilde{C}_{12}|$. With this definition the normalized cross correlation for the noise in expression (1) is given by

$$C_{12}(\omega) = \frac{\langle u(\mathbf{r}_2,\omega)u^*(\mathbf{r}_1,\omega)\rangle}{\langle |u(\mathbf{r}_2,\omega)||u(\mathbf{r}_1,\omega)|\rangle} = e^{ik\hat{\mathbf{n}}\cdot\mathbf{R}_2} \ ,$$
$$C_{13}(\omega) = \frac{\langle u(\mathbf{r}_3,\omega)u^*(\mathbf{r}_1,\omega)\rangle}{\langle |u(\mathbf{r}_3,\omega)||u(\mathbf{r}_1,\omega)|\rangle} = e^{ik\hat{\mathbf{n}}\cdot\mathbf{R}_3} \ . \qquad (3)$$

Note that in general $\langle fg^* \rangle / \langle |fg| \rangle \neq \langle fg^*/|fg| \rangle$, hence the normalized cross correlation is not the same as the cross coherence.

We define a unit vector $\hat{\mathbf{p}}(\varphi)$ such that it has an azimuth φ relative to the vector $\mathbf{R}_2$, see figure 1. The angle φ can be chosen at will, and in the following we integrate over all values of φ. We assume that the three stations are not located along a line. In that case the vectors $\mathbf{R}_2$ and $\mathbf{R}_3$ are independent, and we can write $\hat{\mathbf{p}}(\varphi)$ as a linear superposition of the vectors $\mathbf{R}_2$ and $\mathbf{R}_3$:

$$R_0 \ \hat{\mathbf{p}}(\varphi) = a(\varphi)\mathbf{R}_2 + b(\varphi)\mathbf{R}_3 \ . \qquad (4)$$

The prefactor R_0 ensures that the coefficients $a(\varphi)$ and $b(\varphi)$ are dimensionless. In practice on chooses a value R_0 that is comparable to R_2 and R_3. In the following we suppress in the notation the dependence of the coefficients $a(\varphi)$ and $b(\varphi)$ on the angle φ. Taking the dot product of expression (4) with the vectors $\mathbf{R}_2$ and $\mathbf{R}_3$, respectively, gives using the geometry sketched in figure 1,

$$aR_2 + bR_3\cos\psi = R_0\cos\varphi \ ,$$
$$aR_2\cos\psi + bR_3 = R_0\cos(\varphi - \psi) \ . \qquad (5)$$

This system of equations has the solution

$$a = \frac{R_0}{R_2}\ \frac{\sin(\psi-\varphi)}{\sin\psi} \ \ , \ \ b = \frac{R_0}{R_3}\ \frac{\sin\varphi}{\sin\psi} \ . \qquad (6)$$

We next raise C_{12} to the power a and C_{13} to the power b, with expression (3) this gives

$$C_{12}^a C_{13}^b = e^{ik\hat{\mathbf{n}}\cdot(a\mathbf{R}_2 + b\mathbf{R}_3)} = e^{ikR_0\hat{\mathbf{n}}\cdot\hat{\mathbf{p}}(\varphi)} \ , \qquad (7)$$

where equation (4) is used in the last identity. The right hand side has the same phase as would have been measured by the cross coherence of the wave at two stations with separation $R_0\hat{\mathbf{p}}(\varphi)$. We denote the azimuth of the propagating wave with φ_n, see figure 2, so that equation (7) reduces to

$$C_{12}^a C_{13}^b = e^{ikR_0\cos(\varphi-\varphi_n)} \ , \qquad (8)$$

If we knew the direct of propagation φ_n, we could align the unit vector $\hat{\mathbf{p}}$ with this direction by choosing $\varphi = \varphi_n$ and obtain an unbiased estimate of the velocity c. Since we don't know the direction of propagation, we average equation (8) over all angles φ. Restoring the φ-dependence of the coefficients a and b then gives

$$\frac{1}{2\pi}\int_0^{2\pi} C_{12}^{a(\varphi)} C_{13}^{b(\varphi)} d\varphi = \frac{1}{2\pi}\int_0^{2\pi} e^{ikR_0\cos(\varphi-\varphi_n)} d\varphi \ . \qquad (9)$$

Replacing the integration variable $\varphi \rightarrow \varphi + \varphi_n$ yields

$$\frac{1}{2\pi}\int_0^{2\pi} C_{12}^{a(\varphi)} C_{13}^{b(\varphi)} d\varphi \ = \frac{1}{2\pi}\int_0^{2\pi} e^{ikR_0\cos\varphi} d\varphi$$
$$= J_0(kR_0) \ , \qquad (10)$$

where expression (11.30c) of Arfken and Weber (2001) is used the last identity. Note that expression (10) is independent of the unknown direction φ_n in which the wave propagates.

This approach is similar to the theory derived by Aki (1957) who achieved the angular integration by assuming that uncorrelated noise comes in from all directions with equal amplitude. His ideas formed the basis of what later became known as the SPAC method (Chávez-Garcia and Luzón, 2005; Asten, 2006). One can compute the right hand side of (10) as a function of frequency and tune $k(\omega)$ to fit the measurements. Alternatively one can write the Bessel function as a sum of Hankel functions and get the sum of the causal and acausal Green's function (equations (19.40) and (19.41) of Snieder (2004)).

3 NUMERICAL EXAMPLE

We present a numerical example using the geometry shown in the top left panel of figure 3. The origin is chosen at the location of sensor 1. The x-axis is aligned with $\mathbf{R}_2$. The stations 2 and 3 are at a distance $R_2 = 20$ km, $R_3 = 30.4$ km, and $\psi = 80.5°$. The top panels are computed for random noise propagating with a velocity $c = 3$ km/s coming in at an azimuth $\varphi_n = 130°$, as shown by the red arrow in panel 3a. The noise has a white spectrum containing frequencies between 0 and 5 Hz. In the example we rather arbitrarily used that $R_0 = 19.73$ km. Panel 3b shows the waveforms of expression (8) after a Fourier transform to the time domain, using $R_0 = 19.7$ km. The dependence of the retrieved waveform in the top right panel show a wave arriving with the dependence $\cos(\varphi - \varphi_n)$ predicted by expression (8). The maximum travel time is obtained for $\varphi = 130°$, which corresponds to the direction of the incoming noise. Panel 3c shows that upon integration over φ one obtains causal and anti-causal waves arriving at times $\pm R_0/c = \pm 6.58$ s. The red curve gives the theoretical superposition of waves coming in from all directions. Note that the extracted waveforms are determined by the angles $\varphi = 130°$ and $\varphi = 310°$ where the travel time is stationary. Panel 3a shows the result of integration over φ in the frequency domain. This gives the Bessel function $J_0(kR_0)$ as predicted by expression (10).

The middle panels 3 show the same as the top panels, except that a number of incoming waves, indicated by the red arrows in panel 3d are incident on the receivers. The spread in the direction of the incoming waves is $\pm 5°$. Panel 3d shows that for low wave numbers the Bessel function is reproduced well, but for larger wave numbers the Bessel function is not reproduced. This is reflected in panels 3ef where the high-frequency components of the waves are not reconstructed.

When two noise trains travel in the same direction they behave as one noise train propagating in the same direction. In appendix A we address the question how large the directional difference between two noise trains can be so that they can be treated as propagating in the same direction as measured by the cross correlation. We derive that this is the case when the angle θ between the direction of propagation of the two noise trains satisfies

$$kR_0\, \theta^2 < \frac{\pi}{2} \,. \tag{11}$$

In this example, the waves deviate 5° from the main direction of propagation, to that the angle between the different directions of propagation is 10°. For this angle and the employed parameters, criterion (11) gives $k < 2.6$ km^{-1}. It is close to this wavenumber that the extracted curve in figure 3e deviates from the Bessel function. The lower panels in figure 3 are for a spread of $\pm 10°$ in incoming waves. Now the Bessel function in panel 3g is only reconstructed for the lowest wave numbers. For a difference in propagation of 20°, criterion (11) predicts that only wave numbers $k < 0.7$ km^{-1} are accurately retrieved. This agrees well with the simulation in panel 3g. Since only the lowest wave numbers are retrieved in this case, only the low frequency components or the waves are obtained in panel 3i.

4 DISCUSSION

The theory presented here make it possible to extract the wavenumber and direction of propagation from directional noise. The central idea is that the relative distance between three receivers can be projected onto a vector with arbitrary direction. This projection can be carried out by raising the normalized cross correlations between the stations to prescribed powers. By doing this for all possible directions, and by integrating over these directions, one obtains the wavenumber and direction of propagation of the noise.

The projection of the correlation onto an arbitrary direction is equivalent to placing two receivers at an arbitrary azimuth with respect to each other. The real receivers are, of course, at fixed locations, hence it is as if one creates an array of two receivers that can rotate over an arbitrary angle. The technique thus creates a synthetic rotating aperture to extract the Green's function, hence the name *synthetic aperture Green's function retrieval*.

The concept is based on the assumption that one plane wave propagates through the region of the three receivers. The numerical simulation of section 3 shows that the theory breaks down when this assumption is violated. The criterion (11) predicts how large an angle between propagating noise components can be handled. There are, however, situations where noise is being generated by a single localized source. Examples of such sources include localized industrial activities (Miyazawa et al., 2008), drill bit noise (Poletto and Miranda, 2004), seismic noise generated by a concrete dam (O'Connell, 2007), and, earthquakes.

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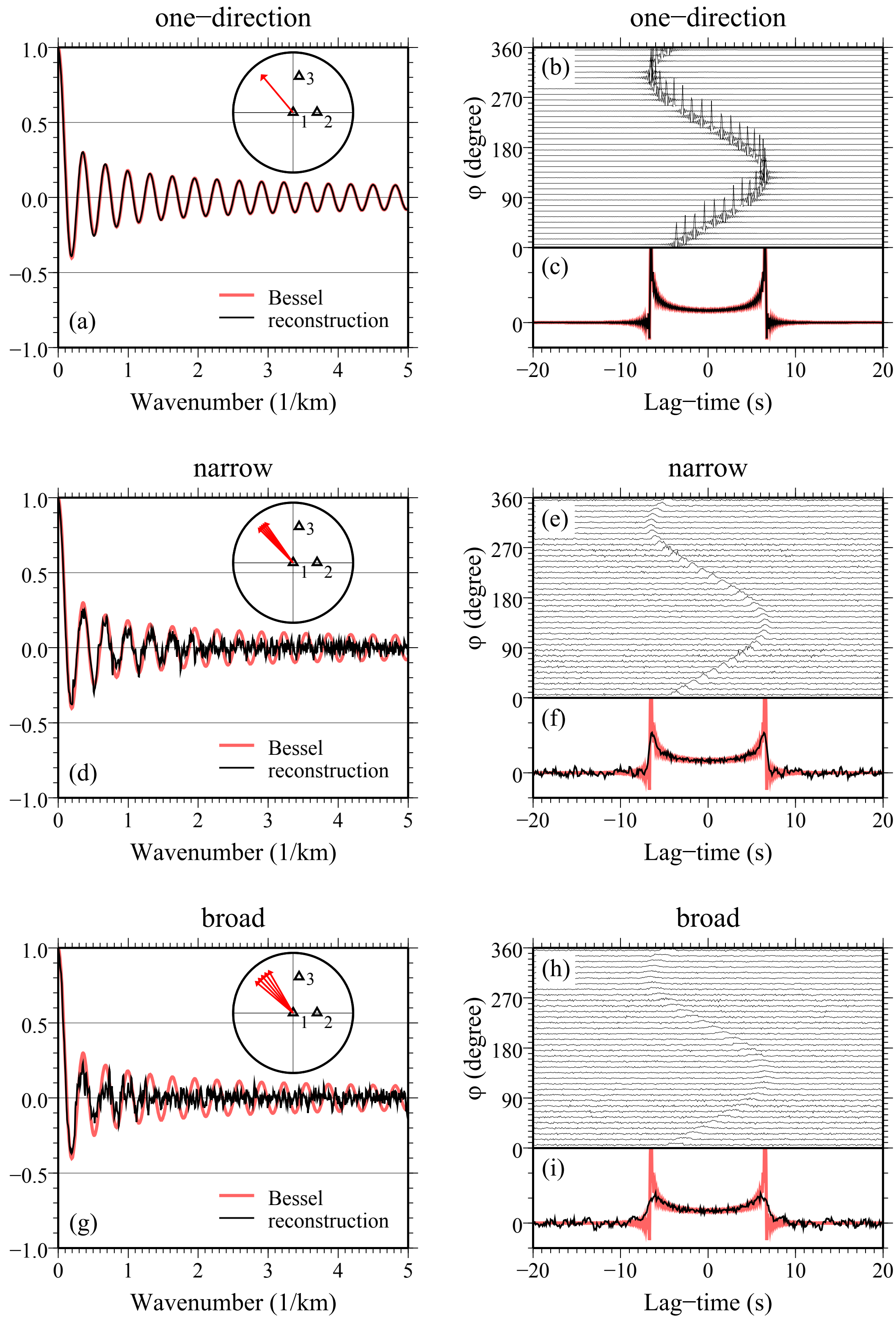

Figure 3. Left panels, the Bessel function $J_0(kR_0)$ (red line) and its reconstruction (black line). the inset in the top right corner gives the directs of the propagating noise (red arrows). Upper right panels, the extracted waves in the time domain as a function of φ, and it's sum over azimuth (black lines in bottom right panels). The red lines in panels c, e, and h, denote the superposition of waves coming in from all directions. Top panels are for one incoming wave, middle panels for waves with a range in azimuth of $\pm 5°$, and bottom panels for waves with a range in azimuth of $\pm 10°$.

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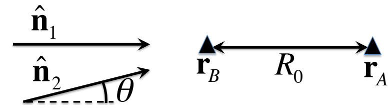

Figure A1. Definition of geometric variables for two incoming waves at a relative azimuth θ.

APPENDIX A: WHEN ARE TWO WAVES DIFFERENT, AS MEASURED BY THE CORRELATION?

When are two waves different as seen by the correlation? Consider the geometry in figure A1 where two receivers are separated over a distance R_0 along the x-axis. One wave comes in along the x-axis, and the second waves comes in as a small angle θ relative to the first wave. The total wave field is given by

$$u(\mathbf{r},\omega) = S_1(\omega)e^{ikx} + S_2(\omega)e^{ik(x\cos\theta + y\sin\theta)} , \qquad \text{(A1)}$$

where y is the coordinate perpendicular to the x-axis. Assuming that the two incoming waves are uncorrelated ($\langle S_1 S_2 \rangle = 0$), the cross correlation of the waves recorded at receivers A and B is given by

$$C_{AB} = \langle u_A u_B^* \rangle = |S_1|^2 e^{ikR_0} + |S_2|^2 e^{ikR_0\cos\theta} , \qquad \text{(A2)}$$

where we used that the receivers are located on the x-axis. The two waves contribute coherently to the cross correlation when the phase difference of the two terms in the right hand side is small compared to a cycle. Assuming that the phase difference must be less than $\pi/4$, this gives the criterion $(kR_0 - kR_0\cos\theta) \leq \pi/4$. For small values of θ this criterion reduces to expression (11).