Tensor-guided fitting of subducting slab depths

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Figure 1. Scattered depth samples (a) from the subducting slab in South America with tensor-guided interpolation (b) of depths accounting for the curvature of the earth’s surface, and tensor-guided fitting (c) of depths accounting for both that curvature and estimated slab strikes. White ellipses represent models of spatial correlation. The solid black lines represent the west coast of South America.

ABSTRACT
Earthquakes and active-source seismic surveys provide estimates of depths of subducting slabs, but only at scattered locations. Constructing a useful 3D model of slab geometry involves fitting the depth estimates with a uniformly sampled function of space. The method used to fit the data should account for the curvature of the earth’s surface as well as data uncertainties. In addition to estimates of depths from earthquake locations, focal mechanisms of subduction zone earthquakes also provide estimates of the strikes of the subducting slab on which they occur. We use estimated strike directions and the earth’s curved surface geometry to infer a model for spatial correlation that guides a blended neighbor interpolation of slab depths. The interpolation method is then modified to account for the uncertainties associated with the depth estimates.

Key words: data fitting, interpolation, subducting slab, tensor-guided, uncertainty
1 INTRODUCTION

Accurate knowledge of the geometry of the interface of a subducting slab has important applications in a variety of seismological analyses such as tsunami modeling and propagation (Wang and He, 2008), seismic hazard assessment (e.g. Global Earthquake Model, http://globalquakemodel.org), tectonic modeling and plate reconstruction, and earthquake source inversions (Hayes and Wald, 2009).

One way of acquiring information about the geometry of a subduction interface is through studying the locations and source mechanisms of the earthquakes that occur on or within the subducting slab. However, earthquakes associated with the subduction process are unevenly distributed. The spatial distribution of such earthquakes is dense in seismogenic sections of the subduction zone and sparse in other areas where seismicity rates are low or nonexistent. For example, shallow sections of subducting slabs (above 80 km) are mostly aseismic. In such areas, in the absence of earthquakes, estimates of slab depths derived from active-source seismic or bathymetry surveys can facilitate a subduction geometry model with higher resolution.

Even after combining earthquake data with bathymetry and active-source seismic surveys, slab depths are sampled only at scattered locations. This situation is illustrated in Figure 1a for the subducting Nazca slab beneath western South America. To construct a useful 3D model of the slab interface, we must interpolate these scattered depths to obtain slab depth as a uniformly sampled function of latitude and longitude.

A large variety of interpolation methods are possible. Several previous studies have attempted to model the slab geometry in subduction zones, particularly in deeper regions below the seismogenic zone. For example, Syracuse and Abers (2006), produced hand-drawn contours to match general Wadati-Beniof Zone structure beneath volcanic arcs; others have produced more generalized multi-regional (Bevis and Isacks, 1984) and global (Gudmundsson and Sambridge, 1998) models.

Hayes et al. (2009) and Hayes et al. (2012) produced Slab1.0 which is a 3D subduction geometry compilation of approximately 85% of the subduction zones worldwide. They used estimates of slab strike derived from earthquake source mechanisms, depths of the earthquakes that those mechanisms represent, and a collection of other complimentary datasets to constrain the geometry of the subduction interface. Their method fits the data with a 3D non-planar surface interpolating between a series of 2D sections which sample the slab geometry every 10 km along the strike of the subduction zone. Each 2D section is constructed using Hermite splines and consists of two parts: a polynomial fit of order 2 or 3 to shallow data (depths≤80 km) splined with a polynomial fit of order 3 or 4 to intermediate and deep data.

The slab geometry model produced by Hayes et al. (2012) is distinguished from its predecessors because it (1) focuses mainly on the shallowest part of subducting slabs while simultaneously attempting to honor the deeper structure of the slab, (2) filters the earthquake data sets used to include only well-located events with thrust mechanisms associated with the subduction process, and (3) includes additional data sets, such as bathymetry and subduction interface interpretations obtained from active-source seismic profiles.

Although Hayes et al. (2012) use strike data to determine the azimuth of the 2D cross sections they fit with splines, they do not incorporate strike data directly in their polynomial fitting.

In this paper, we use the same filtered subduction-related earthquake and active-source seismic data sets as Hayes et al. (2012), but use a different method to produce a 3D model for the subduction interface in South America. Our tensor-guided method is more direct in that we interpolate all of the scattered depth samples at once to obtain slab depth as a function of longitude and latitude. We use the strike directions and the earth’s curved surface geometry to construct a metric tensor field that guides the interpolation of slab depths.

A metric tensor field provides a measure of distance that need not be Euclidean. For the interpolation shown in Figure 1b, the tensor field depicted by the white ellipses describes geodesic distance measured on the curved surface of the earth. The ellipses are elongated because they account for the curvature of the earth’s surface when projected onto an equi-rectangular longitude-latitude coordinate system.

The tensors used to guide the data fitting, shown as white ellipses with variable elongations and orientations in Figure 1c, account for not only the curvature of the earth’s surface but also the estimates of slab strike directions taken from moment tensor and focal mechanism data. The ellipses are more eccentric in areas where the slab dips more steeply. The orientations of the ellipses are determined using the slab strike directions.

The metric tensor fields shown in Figure 1b and 1c guide interpolation (and fitting) by decreasing distances in directions in which the ellipses are elongated, the directions in which slab depths are more highly correlated. These metric tensor fields therefore provide models of spatial correlation that may yield more accurate models of subducting slab interfaces.

Most interpolation methods require, either explicitly or implicitly, a model of spatial correlation for the data to be interpolated. However, some methods for interpolation do not permit spatially varying models of spatial correlation like those depicted in Figure 1. For example, kriging methods in geostatistics do not easily allow for nonstationary anisotropic correlation models (Boisvert et al., 2009). In particular, covariance functions commonly used in kriging are not guaranteed to remain valid (positive definite) when used with...
non-Euclidean distance measures (Curriero, 2006). The blended neighbor interpolation method used in this paper to produce the slab models shown in Figure 1 was developed by Hale (2009) specifically for the purpose of permitting the use of spatially varying models of spatial correlation.

Another distinguishing aspect of our data fitting method is incorporation of uncertainties associated with data. When estimates of such uncertainty are available, the data fitting process must be capable of properly (in a statistical sense) accounting for them. In other words, where significant uncertainty exists, we wish for our model to honor the broad bounds provided by the estimates of uncertainty rather than exactly matching each data point.

We show that the blended neighbor interpolation method (with a slight modification) is capable of accounting for the data uncertainties. This is achieved by altering the smoothness of the blended neighbor interpolation solution so that the prediction errors statistically match the specified uncertainties.

In this paper we first describe the process used to compute metric tensor fields (e.g., the tensors displayed in Figure 1) and then provide the details of our statistical approach for incorporating data uncertainties in our fitting method. Finally, we apply our fitting method to the scattered slab depths shown in Figure 1a to produce a new surface for the subducting Nazca slab beneath South America and compare it with the same surface from Slab1.0 produced by Hayes et al. (2012).

2 THE METRIC TENSOR FIELD

The methodology described in this paper relies on the blended neighbor interpolation method developed by Hale (2009). As shown in appendix A, this method consists of two steps, where each step requires a metric tensor field \( D(x) \) that defines a measure of distance, or equivalently, a model of spatial correlation.

Different tensor fields may be used in different situations. Hale (2009, 2010) shows examples where metric tensor fields are derived from uniformly sampled images. In other situations, where an image is not available, it may be possible to derive the metric tensor fields from other types of secondary data.

In interpolating subducting slab depths, primary data to be interpolated are estimated slab depths; secondary data, from which we derive the tensor field, are estimated slab strikes. The slab depths and strikes are measurements acquired on the curved surface of the earth as a function of longitude and latitude. Therefore, any metric tensor field that we use to guide the interpolation of depths must account for this curvature.

Hale (2011) studies tensor guided interpolation of scattered data on arbitrary non-planar surfaces and provides a general recipe for construction of a tensor field needed for guiding such an interpolation. Here, we follow the same methodology. However, we do not need the same level of generality as provided by Hale (2011). Therefore, in what follows, we will reproduce some of the relevant results presented in Hale (2011) and show the steps of implementing them for addressing the specific problem at hand.

We define the earth’s surface parametrically by a mapping from 2D longitude-latitude coordinates \( u \) to 3D Cartesian coordinates \( x(u) : U \subset \mathbb{R}^2 \rightarrow X \subset \mathbb{R}^3 \), and interpolate slab depths in the 2D space \( U \). In this space each location \( u \) is specified by longitude \( \phi \) and latitude \( \theta \).

The tensor field required to guide a blended neighbor interpolation of depths is constructed in two steps. First, we ignore the curvature of the earth and define a tensor field in an infinitesimal plane tangent to the earth’s surface, where we can assume a flat geometry. Next, we modify this strike tensor field to account for the curvature of the earth.

2.1 Strike tensor field

We base our method for constructing the strike tensor field on the fact that the slab depths should be most highly correlated in the slab strike directions; the spatial correlation of the slab depths is lowest in the slab dip direction (perpendicular to the strike direction). This fact follows intuitively (Figure 2) from the definitions of the strike and dip directions for a dipping plane. Put another way, the dipping structure of the slab results in

![Figure 2. Depths are most highly correlated in the strike direction of the slab. Point D is located somewhere between points A and B in the slab strike direction. In this configuration, although D' is closer to C', the depth at D' is more similar to the depth at A' and B' than it is to the depth at C'. \( \gamma \) denotes the strike angle defined as the azimuth of the strike and measured relative to geographic north N. Dip is perpendicular to the strike direction. \( \delta \) denotes the dip angle measured relative to horizontal.](image-url)
an anisotropic model of spatial correlation for the slab depths. This anisotropy is proportional to the dip angle or slope of the slab; the anisotropy is higher for steeper slopes.

The metric tensor $D$ in the eikonal equation $A3$ is a $2 \times 2$ symmetric and positive definite matrix (Hale, 2009) with two orthonormal eigenvectors $s$ and $d$ and corresponding to positive and real eigenvalues $\lambda_s$ and $\lambda_d$. $D$ can be graphically represented by an ellipse (e.g., see white ellipses in Figure 1) elongated in the direction of the eigenvector corresponding to the maximum eigenvalue and with axes proportional to the square roots of eigenvalues (Hale, 2011). In general, for a tensor field that represents an anisotropic model of correlation, these two eigenvalues are not equal. Here we assume $0 < \lambda_d \leq \lambda_s$.

In the parametric space $U \in \mathbb{R}^2$, the desired strike tensor field $D(u)$ must be designed so that non-Euclidean distances to samples in the strike direction are shorter; such samples therefore get more weight in the interpolation. This design can be achieved by pointing eigenvector $s$ in the slab strike direction and eigenvector $d$ in the slab dip direction (perpendicular to strike), i.e.,

$$s(u) = \begin{bmatrix} \cos \gamma(u) \\ \sin \gamma(u) \end{bmatrix} \quad \text{and} \quad d(u) = \begin{bmatrix} -\sin \gamma(u) \\ \cos \gamma(u) \end{bmatrix}, \quad (1)$$

where $\gamma(u)$ represents the estimated strike angle of the slab (the angle between the estimated strike direction and geographic north) at location $u$ on the earth’s surface.

Now we use eigen-decomposition to construct each tensor $D$ as

$$D = \lambda_s s u s^T + \lambda_d d u d^T, \quad (2)$$

where $s$ and $d$ are the eigenvectors defined in equation 1, and $\lambda_s$ and $\lambda_d$ are their respective eigenvalues. In constructing the strike tensor field for guiding the blended neighbor interpolation of slab depths, what is important is the ratio of the eigenvalues and not their actual sizes; i.e., the aspect ratio of the tensor ellipses is important, not the actual size of their axes. Therefore, we can normalize the eigenvalues and use $\lambda_s(u) = 1$ and some value of $0 < \lambda_d \leq 1$ in equation 2. We let

$$\lambda_d(u) = \frac{1}{1 + \eta \tan^2 \delta(u)}, \quad (3)$$

where $\delta(u)$ is the slab dip angle at location $u$ and $\eta$ is a non-negative real parameter.

Note that with $\lambda_s = 1$ the aspect ratio (eccentricity) of the tensor field ellipses is determined by $\lambda_d$. Therefore, using larger values for $\eta$ amounts to increasing the eccentricity of these ellipses and hence the degree of anisotropy in our model for spatial correlation of slab depths at locations where $\tan(\delta)$, the slope of the slab, is nonzero.

As shown in section 4, a value for the parameter $\eta$ can be determined using a 1D line search and cross-validation. At this point, however, we assume that the value for $\eta$ is known and proceed to the next step which is modifying the strike tensor $D$ in equation 2 to define a metric tensor field that accounts for both strike directions and the curvature of the earth.

### 2.2 Accounting for curvature of the earth

The geodesic distance between two points on a non-planar surface is non-Euclidean. This distance can be calculated by solving an eikonal equation written in the parametric space in which the curved surface is defined (Weber et al., 2008).

In interpolation of subducting slab depths, distances are non-Euclidean because of anisotropy in the model for spatial correlation of depths (inferred from estimated strike directions and the dip angles) and also because of the curvature of the earth’s surface. Therefore, to correctly specify the non-Euclidean measure of distance, the desired metric tensor field must account for a combined effect of slab strikes and curvature of the earth.

We approximate the shape of the earth by a sphere. The gradient operator expressed in spherical coordinates is

$$\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r \cos \theta} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \hat{\theta}. \quad (4)$$

If we denote the radius of the earth by a constant $R$, then on the earth’s surface $r=R$ and equation 4 becomes

$$\nabla = \frac{1}{R \cos \theta} \frac{\partial}{\partial \phi} + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta}. \quad (5)$$

Note that in this equation, $R \cos \theta$ and $R$ are the respective metric scale factors for longitude $\phi$ and latitude $\theta$ in the spherical coordinate system. Equation 5 can be written in matrix form as

$$\nabla = \begin{bmatrix} \frac{1}{R \cos \theta} & 0 \\ 0 & \frac{1}{R \sin \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \theta} \end{bmatrix} \quad (6)$$

or equivalently, following the notation of Hale (2011), as

$$\nabla = F^{-T} \nabla_u, \quad (7)$$

in which $F$ and $\nabla_u$ are defined by

$$F = \begin{bmatrix} R \cos \theta & 0 \\ 0 & R \end{bmatrix} \quad (8)$$

and

$$\nabla_u = \begin{bmatrix} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \theta} \end{bmatrix}. \quad (9)$$

$\nabla_u$ denotes the gradient operator in the parametric longitude-latitude space $U \in \mathbb{R}^2$ defined in section 2. Replacing the gradient operator in eikonal equation $A3$ with the right hand side of equation 7, we can write the
eikonal equation in the parametric space $U$ as
\begin{equation}
\nabla_{u} t^T \ D \ F^{-T} \ \nabla_{u} t = 1.
\end{equation}

In this equation, $D$ is the anisotropic spatial model for correlation (the strike tensor field) defined in equation 2.

In eikonal equation A3, matrix $D$, sandwiched between the two gradients, is the metric tensor. Similarly, in eikonal equation 10, matrix $D_u$, defined as
\begin{equation}
D_u = F^{-1} \ D \ F^{-T},
\end{equation}
is sandwiched between two gradients and therefore can be regarded as the metric tensor in parametric space $U$.

Using $D_u$, we can write equation 10 entirely in parametric space $U$ as
\begin{equation}
\nabla_{u}(t(u)) \cdot D_u(u) \cdot \nabla_{u} t(u) = 1.
\end{equation}

This equation is mathematically similar to equation A3 in every aspect except that it is expressed in the longitude-latitude parametric space $U$. We can use tensor field $D_u$ defined in equation 11 to guide the blended neighbor interpolation of slab depths on the non-planar spherical surface of the earth.

3 INTERPOLATING SLAB DEPTHS

3.1 Data and initial gridding

Primary data used in this study are scattered estimates of depth of the subducting slab in South America (Figure 3a). Each depth estimate is considered to be a random variable with an associated uncertainty (Figure 3b).
Figure 4. Scattered strike data (a) are interpolated on the curved surface of the earth to construct a uniformly sampled strike field (b). The strike field shown in (c) is the result of the 40th iteration of interpolating the scattered strikes guided along the slab dip direction. White ellipses in (b) represent the tensor field $G$ that was used to guide the interpolation on the curved surface of the earth. These ellipses are elongated as they account for the curvature of the earth’s surface when projected onto an equi-rectangular longitude-latitude coordinate system. The white ellipses in (c) are elongated in the slab dip direction and are used to guide the interpolation of strikes in the dip direction.

Following Hayes and Wald (2009), we assume a normal vertical probability density function (pdf) for depth errors. For earthquake data, the half-width (standard deviation $\sigma$) of the normal pdf is chosen based on the location uncertainty reported in the earthquake catalog from which the data were extracted. For active seismic and bathymetry data, the pdf half-width $\sigma$ is based on an assumed uncertainty related to time-to-depth conversion or estimates of sediment thickness (Hayes et al., 2012).

Secondary data consist of scattered estimates of the strike direction of the slab (Figure 4a). The strike data are taken (Hayes and Wald, 2009) from best-fitting double couples of CMT solutions when available (using the gCMT catalog, www.globalcmt.org).

As a first step, we use a uniform rectangular mesh with grid cells of size $0.1^\circ \times 0.1^\circ$ to grid the depth and strike data separately. The value assigned to each grid cell containing bathymetry, active-source seismic, or strike data is the average of all sample values contained in the cell. No values are assigned to the grid cells that contain no strike or depth data.

For earthquake depth data, we use a weighted averaging scheme described in Appendix B, i.e., the depth value and the uncertainty assigned to a grid cell that contains more than one earthquake sample are computed according to equations B2, B4, and B5. This is because the uncertainties associated with earthquake depths are variable and uncorrelated. By using this weighted averaging scheme, we ensure that more certain depth estimates are given more weight than less certain estimates.

Figure 3a shows the location and values (depicted as colors) of the 5076 depth samples after gridding, and Figure 3b shows the uncertainties associated with these samples on a longitude-latitude map. The location and
values of the 458 gridded strike samples are depicted in Figure 4a. The colors represent the estimates of the strike angle $\gamma$ of the slab measured relative to geographic north.

3.2 Interpolation of the strikes

To perform a blended neighbor interpolation of slab depths, we must construct a metric tensor field using the process described in section 2. This tensor field must be defined at each point on the interpolation grid. As shown by equations 1 and 2, to define the metric tensor field $\mathbf{D}$ at each grid point, we need the value of the strike angle $\gamma(u)$ at that point. Therefore, we must first interpolate the scattered strikes.

As shown in Figure 4a, strike data are only available in the region near the coastline. This makes it more difficult to produce accurate estimates of the strike angles for deeper parts of the slab away from the coast.

However, the strikes of the slab are most highly correlated in the dip direction that is perpendicular to the strike direction. This follows from the definition of strike and dip for a dipping plane (Figure 2). Therefore, to obtain more accurate estimates of the strike angles, we guide the interpolation of the scattered strikes in the dip direction using an iterative scheme which consists of three steps:

(i) Interpolate the strikes using tensor-guided interpolation with a metric tensor field that accounts for only the curved geometry of the Earth’s surface. In this case, the non-Euclidean distances are simply geodesic. Therefore, we can use $\mathbf{D} = \mathbf{I}$ in equation 11 to construct a geometry tensor field $\mathbf{G}$ defined as

$$\mathbf{G} = \mathbf{F}^{-1}\mathbf{F}^{-T}. \tag{13}$$

Figure 4b shows the result of the interpolation of scattered slab strikes as a uniformly sampled strike field $\gamma_g(u)$. Tensor field $\mathbf{G}$ used to guide this interpolation is represented by the white ellipses.

(ii) Compute a dip direction field $\gamma_d(u)$ using the strike field $\gamma_g(u)$ as

$$\gamma_d(u) = \gamma_g(u) + 90^\circ \tag{14}$$

because the dip direction is perpendicular to the strike direction.

(iii) Construct a tensor field (white ellipses in Figure 4c) based on $\gamma_d(u)$ and use it to guide a new interpolation of scattered strikes, in the dip direction. The result is a new strike field we call $\gamma(u)$.

This is the end of the first iteration. A new iteration starts by going back to step (ii) but this time, instead of $\gamma_g(u)$ in equation 14, we use $\gamma(u)$ obtained in the previous iteration. This process converges (after 40 iterations) to the strike field shown in Figure 4c.

3.3 Interpolation of the depths

With the strike field $\gamma(u)$ obtained in section 3.2, we define tensor field $\mathbf{D}$ according to equation 2, where we set $\lambda_1 = 1$ and compute $\lambda_2$ using equation 3 with $\eta = 92$. We chose this value for $\eta$ using the procedure described in section 4.

Next we compute $\mathbf{D}_u$, according to equation 11, by multiplying $\mathbf{D}$ on the left by $\mathbf{F}^{-1}$ and on the right by $\mathbf{F}^T$. This is equivalent to combining the strike tensor field $\mathbf{D}$ with geometry tensor field $\mathbf{G}$ defined in equation 13 to get the combined tensor field $\mathbf{D}_u$.

At this point we have all the components needed to perform a blended neighbor interpolation of slab depths on the surface of the earth. Figure 5 shows the results of two blended neighbor interpolations of slab depths guided by two different tensor fields. In Figure 5a, the tensor field $\mathbf{G}$ used to guide the interpolation (shown as white ellipses) accounts for only the curvature of the earth’s surface. In Figure 5b, the tensor field $\mathbf{D}_u$ used to guide the interpolation (shown as white ellipses) accounts for both that curvature and the strike directions.

4 CROSS-VALIDATION

The solution to any interpolation problem is non-unique and depends on the method used for interpolation; the interpolants shown in Figure 5 are just two solutions among infinitely many. These solutions were both obtained using the blended neighbor interpolation method. However the tensor fields used to guide the interpolations were different.

A possible criterion for assessing the results of an interpolation method is to analyze the accuracy of its predictions. We use a 10-fold cross-validation technique (Kohavi, 1995) to find an optimal value for the parameter $\eta$ in equation 3.

In 10-fold cross-validation, input samples are randomly partitioned into ten mutually exclusive subsets (the folds) of approximately equal size (Kohavi, 1995). Subsequently ten iterations of interpolation and validation are performed such that within each iteration a different subset of samples is held out for validation and the union of the remaining 9 subsets (the training set) is used for interpolation.

The validation process involves computing the numerical difference between interpolated values and test-set sample values normalized by the estimates of uncertainty (standard deviation of error) associated with the samples. Based on this normalized difference, a dimensionless error is assigned to each sample in the test set. Therefore, after 10 iterations, a cross-validation error is assigned to all data samples. The accuracy of the interpolation method can then be assessed by analyzing these cross-validation errors. For instance, we can compute the root mean square (rms) of the normalized
errors and use it as a measure of accuracy of an interpolation method.

Now we explain how to use cross-validation to find an optimal value for \( \eta \) in equation 3. The idea is to construct (according to the process described in section 2) different tensor fields using different values for \( \eta \), and compute a separate tensor-guided interpolant using each of these tensor fields. We then compute a cross-validation rms normalized error for each of these tensor-guided interpolants. The optimum value for \( \eta \) is the one corresponding to the interpolant with minimum cross-validation rms normalized error.

One difficulty with this approach is that equation 3 requires knowledge of dip angles of the slab. To compute the dip angles, we smooth the strike ignorant interpolant of Figure 5 in order to obtain an approximate initial slab model \( q_i(u) \). While inaccurate, this initial model is good enough for the purpose of approximating dip angles as

\[
\delta(u) \approx \tan^{-1} \| \nabla q_i(u) \|.
\]
Substituting this approximation into equation 3 yields

$$\lambda_d(u) = \frac{1}{1 + \eta \| \nabla q_i(u) \|^2},$$

which is more straightforward to implement than equation 3.

Figure 6 summarizes the results of the cross-validation process used to determine the best choice for parameter $\eta$. The blue curve shows the cross-validation rms normalized error as a function of $\eta$. The most accurate strike-guided interpolant (the one with the minimum cross-validation rms error) is given by $\eta = 92$.

For a quantitative comparison of the strike-ignorant and strike-guided interpolants shown in Figure 5, we contrast their respective cross-validation rms normalized errors inferred from the curve in Figure 6. The rms normalized cross-validation error for strike-guided interpolation with a tensor field constructed using $\eta = 92$ is approximately 2.82. The same error for the strike-ignorant interpolation guided by a tensor field constructed using $\eta = 0$ is more than three times larger, i.e., 6.34.

5 ACCOUNTING FOR DATA UNCERTAINTIES

The interpolant shown in Figure 5b matches all scattered input depth samples. This is expected as the interpolation error must be (by definition) zero at the location of input samples. However, this interpolant does not represent a geophysically reasonable model for the subduction interface because unlike the interpolant surface of Figure 5b, a model for the slab geometry must be smooth and without abundant inflection points (Hayes and Wald, 2009).

Figure 8a is a vertical cross section showing a profile (green curve) of the interpolated slab depths. Note the abundant fluctuations and excessive curvatures in the green curve. If real, such fluctuations would imply unwarranted bending strains and stresses within the structure of the slab, especially in the shallow parts where the slab is still relatively cold and brittle. Therefore, the fluctuations observed in the interpolated depths are more likely the result of error in depth estimates and do not represent the true geometry of the slab.

To account for data uncertainties and to obtain a smoother model for the slab, we modify the tensor-guided interpolation method to obtain a data fitting method. For this modification, we utilize the given estimates of data uncertainties and incorporate them in a statistically plausible manner into the modeling procedure.

5.1 From interpolation to data fitting

The tensor-guided procedure described so far is based on the blended neighbor interpolation method (summarized in appendix A). The first step of the blended neighbor method can be interpreted as scattering the values $f_k$ corresponding to the nearest known sample points $x_k$ to construct a nearest neighbor interpolant $p(x)$. The second step of the blended neighbor method can be interpreted as smoothing $p(x)$ to obtain a blended neighbor interpolant $q(x)$. The extent of this smoothing at each location $x$ is proportional to the non-Euclidean distance $t(x)$ from $x$ to the nearest known sample (Hale, 2009). This means that at the location of known samples, where $t(x) = 0$, no smoothing is applied so the solution to blending equation A4 is $q(x) = p(x)$; Hence, the interpolation condition $q(x_k) = f_k$ is satisfied and the interpolant $q(x)$ matches all scattered data.

Satisfying the interpolation condition is an implicit requirement of any interpolation method. However, in situations where there is uncertainty associated with data, this condition may not be desirable.

In blended neighbor interpolation, the constraint of satisfying the interpolation condition can be relaxed by adding a function $w(x)$ to the term $t^2(x)$ in equation A4 so that it becomes

$$q(x) = \frac{1}{2} \nabla \cdot (t^2(x) + w(x)) D(x) \cdot \nabla q(x) = p(x).$$

This modification implies that the amount of smoothing applied to $p(x)$ at every location $x$ (including the locations of known samples $x_k$) can be controlled by choosing a proper function $w(x)$. If $w(x_k) \neq 0$ then the solution $q(x)$ of the equation above no longer matches all the scattered samples. In this case, $q(x)$ does not interpolate but, rather, fits the scattered data. Therefore,
instead of interpolation, we shall refer to the modified scheme described above as tensor-guided fitting.

Different fitting functions can be obtained using different smoothing functions \( w(x) \) in equation 19. In practice, we may require a fitting solution that models some real phenomenon. Therefore it is important to choose a \( w(x) \) that results in the desired fitting solution.

For example, in modeling the subducting slab geometry, the uncertainties are not the same for different types of data and different locations (Figure 3b). The estimates of data uncertainties (variance of the errors) are relatively small for shallow sections of the slab (bathymetry and active-source seismic data) and larger for deeper sections of the slab (earthquake data).

Therefore, the smoothing function \( w(x) \) must be designed so that the amount of smoothing applied to the location of each data sample is proportional to specified data uncertainties. One such smoothing function can be defined as

\[
w(x) = s \sigma^2(x),
\]

where \( s \) is a positive scalar parameter and \( \sigma(x) \) is a smooth model of standard deviation that approximates the actual standard deviation of the error associated with data. This model of standard deviation is shown in Figure 7.

Using \( w(x) \) defined above in equation 17, we obtain

\[
q(x) - \frac{1}{2} \nabla \cdot \left( t^2(x) + s \sigma^2(x) \right) D(x) \cdot \nabla q(x) = p(x).
\]

The smoothing parameter \( s \) in equation 19 controls the smoothness of the fitting solution \( q(x) \). Note that tensor-guided interpolation is a special case (with \( s = 0 \)) of the more general tensor-guided fitting (with \( s > 0 \)).

The question left to be answered is, therefore, how to choose the smoothing parameter \( s \).

### 5.2 Choosing the smoothing parameter

In real problems, there are often constraints that make one fitting solution preferable to other possible solutions. For instance, in constraining the geometry of a subduction slab, a geophysically reasonable model for the slab is expected to be smooth. Nevertheless, such a model must not be so smooth that sampled depths are completely disregarded.

Therefore, to obtain an optimum fitting function, we must find a balance between the smoothness of \( q(x) \) and the degree to which it honors the information given by data. Before explaining how to find this balance, we define some new terms.

We define the standardized data error (error in each depth sample value) as

\[
\hat{e}_k = \frac{\mu_k - f_k}{\sigma_k}
\]

where \( k \) is the sample index, \( \mu_k \) is the expected value of slab depth (true depth) at location \( x_k \) of the sample, \( f_k \) is the sample value, and \( \sigma_k \) is the uncertainty or the standard deviation of the error associated with the sample. Note that in equation 20, \( \mu_k \) and \( \hat{e}_k \) are unknown quantities.

Recall that each depth sample is assumed to be a random variable with a normal distribution \( N(\mu_k, \sigma_k) \). Therefore, the standardized data error \( \hat{e}_k \) is expected to be a random variable with standard normal distribution. From this, we infer that the collection of all standardized data errors \( \hat{E} \), computed according to equation 20, also constitutes a population that is a standard normal distribution, i.e.,

\[
\hat{E} \sim \mathcal{N}(0, 1).
\]

Similarly, we define the standardized fitting error

\[
\hat{E} \sim \mathcal{N}(0, 1).
\]
(residual) at location $\mathbf{x}_k$ of each sample as

$$\hat{r}_k = \frac{q(\mathbf{x}_k) - f_k}{\sigma_k} \tag{22}$$

where $k$ is the sample index, $q(\mathbf{x}_k)$ is the value of the fitting function at location $\mathbf{x}_k$ of the sample, $f_k$ is the sample value, and $\sigma_k$ is the uncertainty or standard deviation of the error associated with the sample.

Our goal is to find a fitting function $q(x)$ that correctly estimates the true slab depths. If this goal is attained for some optimum fitting function $q_{op}(x)$, then we have $q_{op}(\mathbf{x}_k) = \mu_k$ and hence, by equations 20 and 22, $\hat{r}_k = \hat{e}_k$. This implies that, for the optimum solution, the collection of all standardized fitting errors $\hat{R}$ will have a standard normal distribution, i.e.,

$$\hat{R}|_{q_{op}(x)} \sim \mathcal{N}(0, 1). \tag{23}$$

This can be used as a criterion to choose the optimum smoothing parameter $s$ in our data fitting method. In other words, we can analyze fitting errors associated with fitting solutions computed using different smoothing parameters, and then choose the one for which $\hat{R} \sim \mathcal{N}(0, 1)$ as the optimal smoothing parameter $s$.

Note that our assumption about the normality of the distribution of the fitting errors might not be accurate. Therefore, when assessing the distribution of the fitting errors, it is important to use a robust statistic which is not severely affected by the potential outliers. One such statistic is the interquartile range (IQR). For $\mathcal{N}(0, 1)$, the IQR is the range of values from -0.674 to 0.674 containing 50% of the population. Thus, for an optimum fitting solution, half of the standardized fitting errors are expected to lie within the range $[-0.674, 0.674]$.

To find the right smoothing parameter, we start from the interpolation solution (i.e., from $s = 0$) and then gradually increase the smoothness parameter until we reach a point where half of the fitting errors fall within and half fall outside of the IQR for a standardized normal distribution. We choose the smoothing parameter $s$ for which this condition is satisfied to be the optimal fitting parameter and use it to compute the optimum fitting solution.

### 6 RESULTS AND DISCUSSION

The final result of applying our tensor-guided fitting procedure to slab data is shown in Figure 8c. Compared to the interpolating model (shown in Figure 8b), this fitting model is smoother and is therefore likely to be more geophysically reasonable. The smoothing parameter $s = 0.56$, employed in the tensor-guided fitting to produce the model in Figure 8c was chosen using the method described in section 5.2.

The fitting solution obtained by Hayes et al. (2012) in the Slab1.0 subduction slab compilation is shown in Figure 8d. A cross-section profile of Slab1.0 is contrasted with the tensor-guided fitting and interpolating profiles in Figure 8a. In this cross section, the Slab1.0, the tensor-guided fitting, and the tensor-guided interpolating profiles are the blue, red, and green curves, respectively. Compared to the tensor-guided interpolation, the tensor-guided fitting profile is clearly smoother.

However, our fitting profile (red) is not as smooth as the is Slab1.0 profile (blue). This is because Hayes et al. (2012) use polynomial splines of degrees 2, 3, or 4 to fit the data in 2D sections. Therefore, Slab1.0 solution is forced to be smoother than the tensor-guided fitting result in the cross section shown here. Nevertheless, our model honors the data more precisely than the Slab1.0 model.

Also, note that the 3D slab surface produced by our fitting method (Figure 8c) is smoother than the Slab1.0 model (Figure 8d) along the direction parallel to the coastline. One reason for this is that unlike the method used by Hayes et al. (2012) which requires interpolating between interpolated 2D profiles along the coastline, tensor-guided fitting is performed in one step. Another reason is that the tensor-guided fitting solution is guided along the strike directions of the slab.

The difference between the depths predicted by the tensor guided method and the depths predicted by the Slab1.0 model is in most areas 0 – 40 km. The tensor-guided fitting and interpolating solutions change concavity with their slopes approaching horizontal at the eastern edge of the model (see Figure 8a around distance 900-1070 km and depth 400-600 km). This is due to the use of a zero-slope boundary condition in solving the system of partial differential equations 19.

Figure 9 shows the histogram of standardized fitting errors $\hat{r}_k$ associated with the optimum fitting solution computed using smoothing parameter $s = 0.56$. Note that this histogram does not exactly represent a normal distribution. The long tails of the distribution observed here are not expected for a normal distribution. Therefore, this study might benefit from assuming a different type of distribution (e.g., exponential) for the data and fitting errors.

Also, note that the histogram shown in Figure 9 is not centered at 0, which indicates a bias towards positive fitting errors. As explained below, the observed bias is a consequence of the geometry of the problem, and the smoothing applied in our fitting procedure.

The slab geometry is almost horizontal in areas near the coastline. Therefore, the tensors used in the tensor-guided fitting procedure in these areas have low eccentricity (e.g., see the nearly circular ellipses to the left of the coastline in Figure 5). One consequence of using these tensors in the smoothing step is that the fitting solution $q(x)$ in the bathymetry area receives contribution from the deeper sections of the slab. Thus, $q(x_k)$ is more likely to overestimate the slab depth at locations $x_k$ of the bathymetry samples. This means that the stan-
Figure 8. A cross section (a) showing the profiles of three different slab models. The slab model obtained by tensor-guided interpolation (b) is compared with the model obtained by tensor-guided fitting (c) and the same model from Slab1.0 (d) produced by Hayes et al. (2012). Line segment AB shows the geographical location of the vertical cross section shown in (a). The gray crosses in (a) are the orthogonal projection of all data points (the points shown in white in (b), (c), and (d)) that lie within a rectangular window of width 100 km centered on the vertical plane of section AB. The gray dots in (b), (c), and (d) denote the location of scattered data points.
Tensor-guided fitting of subducting slab depths

standardized fitting errors defined in equation 22 are more likely to be positive than negative for the bathymetry data. Recall that about 40% of the depth samples in this study are bathymetry data densely clustered near the coastline (Figure 3). Therefore, the positive fitting errors statistically outnumber the negative fitting errors and a positive bias in the distribution of errors is created.

7 CONCLUSION

Additional information provided by secondary data can be used to guide the interpolation or fitting of a primary set of spatially scattered data. This is done through construction of a metric tensor field that defines a model for spatial correlation. The tensor-guided fitting method presented here is capable of utilizing tensors that are both anisotropic and spatially variable.

We constructed a tensor field based on the earth’s curved surface geometry and strike estimates of the subducting slab in South America and used it to guide the interpolation of the scattered estimates of slab depths.

Proper handling of data uncertainties is an important aspect of data fitting methods. If estimates of uncertainties associated with data are available, they can be easily incorporated into the tensor-guided fitting procedure to produce a statistically consistent fit to the scattered data. We used the slab data to demonstrate the capability of our data fitting method in handling data uncertainties.

In other areas of geoscience, such as exploration geophysics, atmospheric, oceanic, and environmental studies, there are problems and applications similar to the one considered in this paper, where the spatial correlation model for one scattered dataset can be inferred from another and where estimates of uncertainty associated with data are available. Therefore, the process of constructing a metric tensor field and accounting for the uncertainties described here can serve as an example for applying tensor-guided fitting in such problems.

8 ACKNOWLEDGMENTS

First author wishes to thank Simon Luo for the helpful discussions on this paper and Diane Witters for her instructions on revision and editing. This research was supported by the sponsors of the Center for Wave Phenomena at the Colorado School of Mines.

APPENDIX A: BLENDED NEIGHBOR INTERPOLATION

Blended neighbor interpolation is specifically designed to facilitate tensor-guided interpolation of scattered data (Hale, 2010).

If we assume the scattered data to be interpolated are a set

$$\mathcal{F} = \{f_1, f_2, \ldots, f_K\}$$

of K known sample values \( f_k \in \mathbb{R} \) corresponding to a set

$$\mathcal{X} = \{x_1, x_2, \ldots, x_K\}$$

of K known sample points (coordinates) \( x_k \in \mathbb{R}^n \), then the goal of interpolation is to use the known samples to construct a function \( q(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( q(x_k) = f_k \).

In the blended neighbor interpolation method (Hale, 2009), the interpolant \( q(x) \) is constructed in two steps:

(i) Solve the eikonal equation

$$\nabla t(x) \cdot D(x) \cdot \nabla t(x) = 1, \quad x \notin \mathcal{X};$$

$$t(x_k) = 0, \quad x_k \in \mathcal{X}$$

for \( t(x) \): non-Euclidean distance from \( x \) to the nearest known sample \( x_k \), and \( p(x) \): the value \( f_k \) of the sample at point \( x_k \) nearest to the point \( x \).

(ii) Solve the blending equation

$$q(x) - \frac{1}{2} \nabla \cdot \nabla t^2(x)D(x) \cdot \nabla q(x) = p(x)$$

for the blended neighbor interpolant \( q(x) \).

In the equations above, \( D \) is a metric tensor field which defines the measure of distance in space by providing the anisotropic and spatially varying coefficients of the eikonal equation. In \( n \) dimensions, the metric tensor field \( D \) is a symmetric and positive definite \( n \times n \) matrix (Hale, 2009).
APPENDIX B: COMBINING MEASUREMENTS HAVING RANDOM UNCORRELATED ERRORS AND KNOWN VARIANCES

Here, we discuss a way to combine independent measurements of a quantity using a weighted averaging scheme.

Consider $N$ independent measurements $(x_1, x_2, ..., x_N)$ of the same quantity where each measurement has an unknown expected value and error, i.e.,

$$x_i = \mu + e_i, \quad i = 1, 2, ..., N$$  (B1)

where $\mu$ is the true value of the quantity we wish to estimate, and $e_i$ is the error in each measurement.

We assume that the errors are not correlated, their expected values are zero, and their variances $\sigma_i^2$ are known.

We let the combined measurement $x$ to be a weighted average (linear combination) of the individual measurements $x_i$,

$$x = \sum_{i=1}^{N} w_i x_i,$$  (B2)

requiring that the weights $w_i$ must satisfy the unbiasedness condition

$$\sum_{i=1}^{N} w_i = 1.$$  (B3)

Using these assumptions, the variance of the combined measurement can be expressed as

$$\sigma^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2.$$  (B4)

By minimizing this variance with respect to $w_i$ (subject to the unbiasedness constraint of B3) the weights $w_i$ can be determined as

$$w_i = \frac{1}{\sigma_i^2} \sum_{j=1}^{N} \frac{1}{\sigma_j^2}, \quad i = 1, 2, ..., N.$$  (B5)

Using these weights, we can compute the best (minimum variance) estimate for the combined measurement $x$ and its variance $\sigma^2$ using equations B2 and B4, respectively. Note that to arrive at these results, we assumed no particular statistical distribution for the measurement errors.

REFERENCES


Hayes, G. P., and D. J. Wald, 2009, Developing framework to constrain the geometry of the seismic rupture plane on subduction interfaces a priori - a probabilistic approach: Geophys J. Int, 176, 951–964.


Hayes, G. P., D. J. Wald, and K. Keranen, 2009, Advancing techniques to constrain the geometry of the seismic rupture plane on subduction interfaces a priori: Higher-order functional fits: Geochemistry Geophysics Geosystems, 10, 1–19.


