Elastic waves as a Foucault pendulum?

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ABSTRACT
The rotation of the Earth leads to a change in the equation of motion because of the Coriolis force. We study how Earth's rotation affects elastic wave propagation. We show that the Coriolis force does not deflect seismic rays because the medium that carries the waves moves with Earth's rotation. However, Earth's rotation causes a weak traverse component of P-waves and a weak longitudinal component for S-waves. More importantly, S-waves with opposite circular polarizations propagate with different velocities. These two circularly polarized S-waves can be superposed to form a lineally polarized shear wave whose polarization slowly rotates in exactly the same way as Foucault's pendulum. When Earth is rotated over 2π the polarization of the shear waves does not return to its original value, unless the wave propagates along the rotation axis. This is a seismic manifestation of the Berry phase.

Key words: Seismology, elastic wave propagation, Earth rotation, Berry phase

1 INTRODUCTION

In this paper we analyze the imprint of Earth's rotation on seismic wave propagation. This research was spurred by beam forming of long-period body waves as shown in figure 1. Our original motivation for doing the beam forming was to study the scattering properties of long-period body waves that propagate deep into the Earth and that were recovered effectively from seismic interferometry (Lin et al., 2013; Lin and Tsai, 2013; Boué et al., 2013, 2014).

We used data from a deep earthquake (depth of 606 km) of magnitude $M_w = 8.3$ in the Sea of Okhotsk from May 24, 2013 that were recorded on stations of USArray (figure 1). In this example we bandpass filtered the data between periods of 32 s and 48 s. Because of the long periods the waves do not attenuate quickly and the waves excited by the earthquake stand out above the noise for about half a day after the earthquake.

The wave field across USArray for the direct P-wave is shown in figure 1a along with the great circle (green line) that connects the earthquake to the center of the array. At this scale the wavefield is almost plane, and propagates in the direction of the great circle. This wavefield is after beam forming shown in panel 1b as a function of horizontal slowness (s_x, s_y). The width of dot in panel 1b corresponds to the array response of the beam forming. This dot has a slowness that matches the orientation of the great circle well.

Panels 1c-e show the beam forming averaged over time windows that are indicated above each panel. In panel 1c the waves arrive in the opposite direction of the waves panel 1b for the direct P-wave. The reason is that most waves arriving between 2-3 hours after the earthquake propagate along the major arc instead of the minor arc, and thus propagates in the opposite direction along the great circle. Panel 1e is for waves that have propagated between 8-9 hours. Note that for these late times most of the energy still propagates along the great circle. Apparently, at these long periods (32 s to 48 s), scattering is so weak that the field is not directionally homogenized in 8 hours.

In 8 hours, Earth has rotated over 120°. Yet in panel 1e, the energy still arrives along the great circle path. One might expect that seismic waves are deflected by the Coriolis force. This is not a real physical force, but a pseudo-force that is an artifact from the fact the a rotating coordinate frame is accelerating (Snieder and van Wijk, 2015). Shouldn't seismic rays be influenced by the Coriolis force? Do seismic waves that bounce back in forth in the Earth precess in a way that is similar as a Foucault pendulum? Figure 1e suggests that this is not the case because a deflection over 120°, Earth's rotation in 8 hours, should be detectible by the beam forming.

In this paper we explore the imprint of Earth's rota-

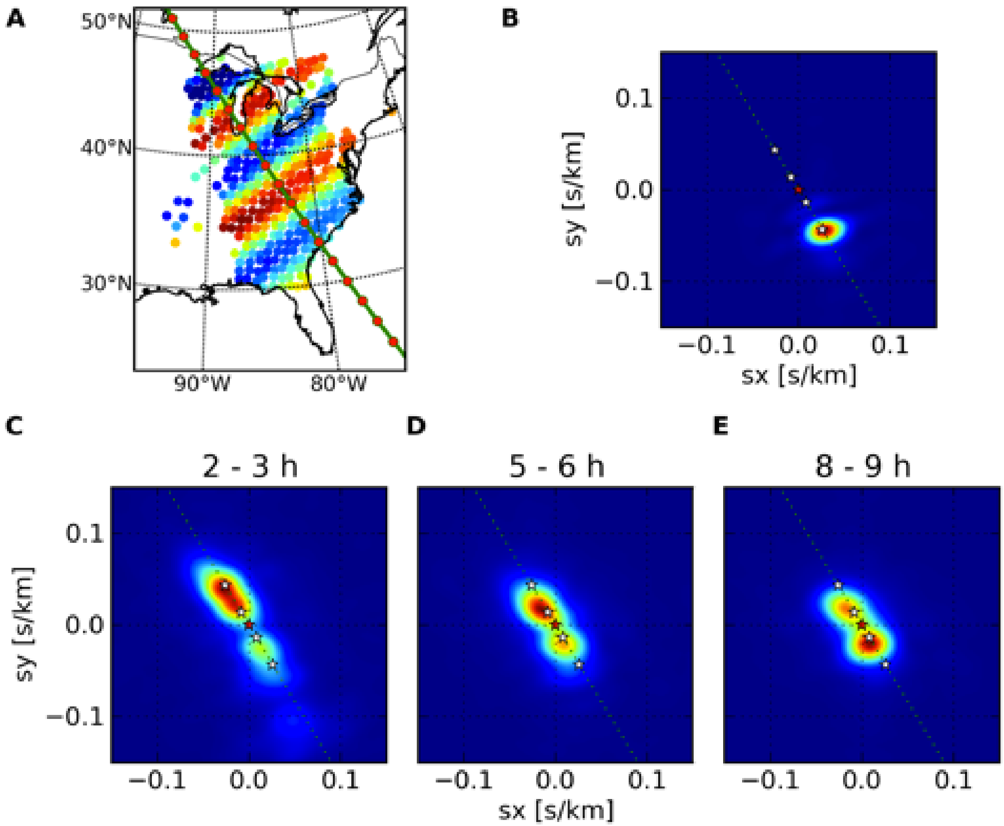

Figure 1. Panel a: wavefield, bandpass filtered for periods between 32 s and 48 s recorded at US Array after a deep earthquake in the Sea of Okhotsk. The great circle from the earthquake to the center of the array is shown by the green line. Panels b-e: beam forming of the wavefield recorded on the array in panel a using waves in different time intervals after the earthquake as indicated above each panel, except for panel b which is computed for the direct P-wave. The horizontal and vertical axes show slowness. The stars denote arrivals of main body wave phases that propagate along the great circle.

tion on wave propagation. Figure 1e suggests that rays are not deflected by the Coriolis force, but it is known that Earth's normal modes are influenced by rotation (Backus and Gilbert, 1961). Therefore there is some imprint of Earth's rotation on seismic waves. Schoenberg and Censor (1973) analyzed the propagation for elastic waves in homogeneous anisotropic media for an arbitrary rotation rate, but the physical implications of the rotation are lost in the mathematical complexity of that work. In this work we restrict attention to isotropic media and restrict ourselves to low rotation rates Ω compared to the angular frequency ω of the seismic waves. We show in section 2 why rays are not deflected by the Coriolis force. In section 3 we derive and analyze the Christoffel equation for plane elastic waves in a rotating system and show how the polarization of P-and S-waves is affected by the rotation. We next analyze in section 4 how the polarization of S-wave is changed by Earth's rotation. The derivation of the polarization of elastic waves in rotating media is shown in an appendix.

2 ARE RAYS DEFLECTED BY EARTH'S ROTATION?

We investigate in this section the imprint of Earth's rotation on seismic rays. To this end we consider rays whose direction is denoted by the unit vector $\hat{\mathbf{n}}$. The equation of kinematic ray tracing is given by equation (4.44) of Aki and Richards (2002)

$$\frac{d}{ds}\left(\frac{1}{c}\hat{\mathbf{n}}\right) = \nabla\frac{1}{c}\ , \tag{1}$$

where c is the wave velocity and s the arc length along the ray. The gradient in the right hand side can be decomposed in a component along the ray and a perpendicular component: $\nabla = \hat{\mathbf{n}} d/ds + \nabla_\perp$. Using this decomposition, expression (1) can be written as $d\hat{\mathbf{n}}/ds = c\nabla_\perp(1/c)$. An increment ds along the ray corresponds to an increment in travel time $dt = ds/c$, so that the equation of kinematic ray tracing can also be written as

$$\frac{d\hat{\mathbf{n}}}{dt} = c^2 \nabla_\perp \frac{1}{c} \ . \tag{2}$$

We next consider a rotating Earth and analyze the equation of kinematic ray tracing in a coordinate system that does not rotate. In that fixed coordinate system, there is no Coriolis force, but according to expression (12.22) of Snieder and van Wijk (2015), the medium rotates with a velocity

$$\mathbf{V} = \boldsymbol{\Omega} \times \mathbf{r} \ , \tag{3}$$

where $\boldsymbol{\Omega}$ is Earth's rotation vector. According to expression (3) of Kornhauser (1953), wavefronts in a medium that moves with velocity $\mathbf{V}$ satisfy the eikonal equation with velocity

$$c_M = c + (\hat{\mathbf{n}} \cdot \mathbf{V}) \ . \tag{4}$$

For this velocity the gradient is given by

$$\nabla \left(\frac{1}{c_M} \right) = \frac{1}{c_M^2} \left(c^2 \nabla \frac{1}{c} - \nabla(\hat{\mathbf{n}} \cdot \mathbf{V}) \right) \ . \tag{5}$$

Using this in expression (2) for the velocity c_M gives

$$\frac{d\hat{\mathbf{n}}}{dt} = c^2 \nabla_\perp \frac{1}{c} - \nabla(\hat{\mathbf{n}} \cdot \mathbf{V}) \ . \tag{6}$$

The first term accounts for the ray bending due to the velocity gradient, while the second term accounts for the ray deflection due to the movement in the medium. For the rigid rotation (3), $(\hat{\mathbf{n}} \cdot \mathbf{V}) = n_x \Omega_y z - n_x \Omega_x y + n_y \Omega_z x - n_y \Omega_x z + n_z \Omega_x y - n_z \Omega_y x$, hence

$$\nabla(\hat{\mathbf{n}} \cdot \mathbf{V}) = -\boldsymbol{\Omega} \times \hat{\mathbf{n}} \ . \tag{7}$$

Inserting this in the equation of kinematic ray tracing for the moving medium (6) gives

$$\frac{d\hat{\mathbf{n}}}{dt} = c^2 \nabla_\perp \frac{1}{c} + \boldsymbol{\Omega} \times \hat{\mathbf{n}} \ . \tag{8}$$

Note that both terms are perpendicular to the unit vector $\hat{\mathbf{n}}$, so that it's norm is preserved.

According to equation (12.18) of Snieder and van Wijk (2015), the last term describes the co-rotation of $\hat{\mathbf{n}}$ with Earth's rotation. The rays thus co-rotate with Earth's rotation, which means that in a coordinate system that rotates with the Earth the rays are not deflected. The Coriolis force thus does not deflect rays. Physically this corresponds to the fact that rays are not trajectories of material particles that are subject to the inertia causing the Coriolis force. We next analyze the polarization of P- and S-waves in a rotating coordinate system.

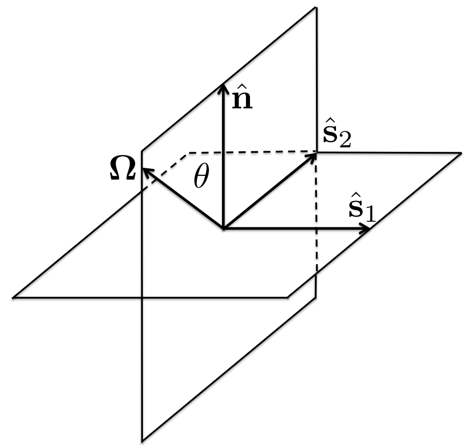

Figure 2. Definition of the unit vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$.

3 PLANE ELASTIC WAVES IN A ROTATING SYSTEM

We consider the special case of a homogeneous isotropic elastic medium and assume that that rotation rate is small, in the sense that $\Omega/\omega \ll 1$, where ω is the angular frequency of the waves. For the extreme case of Earth's gravest mode (Stacey, 1992) $\Omega/\omega \approx 1\ hour/1\ day \approx 0.04$, which is indeed much smaller than 1.

The equation of motion in an homogeneous elastic medium follows from expression (4.1) of Aki and Richards (2002)

$$\rho\ddot{\mathbf{u}} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} - 2\rho\boldsymbol{\Omega} \times \dot{\mathbf{u}} \ , \tag{9}$$

where ρ is the mass density, λ and μ the Lamé parameters, and the overdot denotes a time-derivative. Note that we have left out the centrifugal force $-\rho\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$, the corrections caused by this term are of the second order $(\Omega/\omega)^2$, which we ignore in the following. We seek solutions of the form

$$\mathbf{u} = \hat{\mathbf{q}} e^{i(k\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t)} \ , \tag{10}$$

where the unit vector $\hat{\mathbf{n}}$ gives the direction of propagation and $\hat{\mathbf{q}}$ the polarization. Inserting this in expression (9) gives the Christoffel equation

$$\hat{\mathbf{q}} = \frac{\lambda + \mu}{\rho c^2} \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{q}}) + \frac{\mu}{\rho c^2} \hat{\mathbf{q}} - \frac{2i}{\omega} \boldsymbol{\Omega} \times \hat{\mathbf{q}} \ , \tag{11}$$

where $c = \omega/k$ the wave velocity.

We next define three orthogonal unit vectors to describe the polarization as shown in figure 2. The first unit vector $\hat{\mathbf{n}}$ is the direction of wave propagation. The second unit vector $\hat{\mathbf{s}}_1$ is defined to be perpendicular to $\hat{\mathbf{n}}$ and $\hat{\boldsymbol{\Omega}}$, hence it is proportional to $\hat{\mathbf{n}} \times \hat{\boldsymbol{\Omega}}$. The third unit vector $\hat{\mathbf{s}}_2$ is perpendicular to the two other unit vectors and is defined as $\hat{\mathbf{s}}_2 = \hat{\mathbf{n}} \times \hat{\mathbf{s}}_1$. The unit vectors $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ are thus given by

$$\hat{\mathbf{s}}_1 = \frac{1}{\sin\theta} \ \hat{\mathbf{n}} \times \hat{\boldsymbol{\Omega}} \ \ , \ \ \hat{\mathbf{s}}_2 = \frac{1}{\sin\theta} \left(\cos\theta \ \hat{\mathbf{n}} - \hat{\boldsymbol{\Omega}} \right) \ , \tag{12}$$

where θ is the angle between the direction of wave propagation and the rotation vector, as shown in figure 2. The vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$ form a right-handed system. We derive the polarization vectors for P and S-waves in appendix A, and summarize the results below.

For the P-waves the polarization vector is given by

$$\hat{\mathbf{q}}_P = \hat{\mathbf{n}} + \frac{2i\Omega \sin\theta}{\omega} \frac{\lambda + 2\mu}{\lambda + \mu} \hat{\mathbf{s}}_1 \ , \tag{13}$$

and the wave velocity is given by

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} + O(\Omega/\omega)^2 \ . \tag{14}$$

The last expression simply states that the wave velocity of P-waves is to first order in Ω/ω not affected by Earth's rotation. According to equation (13) the polarization is not purely in the direction of propagation; the P-wave has a small transverse component. This transverse component is 90 degrees out of phase with the longitudinal component, so the P-wave is elliptically polarized with eccentricity proportional to $\Omega \sin\theta/\omega$. This ellipticity is caused by the sideways deflection of the particle motion by the Coriolis force, and it depends on the angle θ between the direction of propagation and the rotation vector. The ellipticity vanishes when the wave propagates along the rotation axis ($\theta = 0$) and is largest when the wave propagates perpendicular to the rotation axis ($\theta = \pi/2$). The ellipticity depends on frequency and is most pronounced for low frequencies ω. Note that the polarization (13) depends on the Lamé parameters, this is due to the elastic restoring forces that counteract the Coriolis force.

For the S-waves there are two solutions that are both predominantly circularly polarized, the polarization and propagation velocity of the two shear wave solutions is given by

$$\begin{aligned} \hat{\mathbf{q}}_{S+} &= \frac{1}{\sqrt{2}}(\hat{\mathbf{s}}_1 - i\hat{\mathbf{s}}_2) + \frac{a}{\sqrt{2}}\hat{\mathbf{n}} \quad \text{with} \quad c_S = \sqrt{\frac{\mu}{\rho}} + \delta c_S \ , \\ \hat{\mathbf{q}}_{S-} &= \frac{1}{\sqrt{2}}(\hat{\mathbf{s}}_1 + i\hat{\mathbf{s}}_2) + \frac{a}{\sqrt{2}}\hat{\mathbf{n}} \quad \text{with} \quad c_S = \sqrt{\frac{\mu}{\rho}} - \delta c_S \ , \end{aligned} \tag{15}$$

where the velocity shift δc_S is given by

$$\frac{\delta c_S}{c_S} = \frac{\Omega}{\omega} \cos\theta \ . \tag{16}$$

This shift in the shear velocity is similar to the frequency shift $\delta\omega_n$ of Earth's normal modes caused by the Coriolis force derived by Backus and Gilbert (1961) who show for toroidal modes that

$$\frac{\delta\omega_n}{\omega_n} = \frac{\Omega}{\omega_n} \frac{m}{l(l+1)} \ , \tag{17}$$

where ω_n is the normal mode frequency for mode n, while l and m are the angular order and degree, respectively. For spheroidal modes a similar expression holds, but the right hand side contains a dimensionless constant that depends on the modal structure. Note $m/l(l+1)$ is equation (17) plays the same role as $\cos\theta$ in expression (16), because m is the z-component of the angular momentum of the spherical harmonics while $l(l+1)$ is the total angular momentum (Merzbacher, 1970).

The coefficient a in expression (15) is given by

$$a = \frac{2i\Omega \sin\theta}{\omega} \frac{\mu}{\lambda + \mu} \ . \tag{18}$$

Hence the terms $a\hat{\mathbf{n}}$ in equation (15) denote that the S-waves have a slight elliptical polarization in the longitudinal direction that is akin to the slight elliptical polarization of the the P-wave. The frequency dependence of δc_S implies that the S-waves are dispersive. As we show in the next section, this frequency dependence accounts for a constant rate of rotation of the S-wave polarization in the plane perpendicular to the direction of wave propagation.

According to expression (18), the coefficient a is $O(\Omega/\omega)$, and is thus small compared to the terms $\hat{\mathbf{s}}_1 \pm \hat{\mathbf{s}}_2$ in equation (15). For this reason we ignore the longitudinal component $a\hat{\mathbf{n}}/\sqrt{2}$ in the following. The two shear waves in expression (15) are circularly polarized and move with slightly different velocities. This behavior is reminiscent of the Bravais pendulum (Babović and Mekić, 2011). Just as the Foucault pendulum, the Bravais pendulum consists of a mass that is suspended by a long thin wire. In Foucault's pendulum the mass oscillates in a plane, while in the Bravais pendulum the mass moves in a a circular orbit, either in the clockwise or in the counterclockwise direction. When this orbit moves against Earth's rotation it takes less time to move over a full circle than when it moves with Earth's rotation. The difference in the orbit times for the two circular motions of the mass can be used to measure Earth's rotation. According to expression (15) the two circularly polarized shear waves behave similarly as the Bravais pendulum.

When observing S-waves, they typically are linearly polarized. The motion of two circular polarizations can be superposed to give a linear polarization. In the next section we superpose the two circularly polarized shear waves to produce a linearly polarized shear wave.

4 THE ROTATING POLARIZATION OF S-WAVES

In order to study the polarization of the S-waves in the transverse plane, we ignore the terms involving $a\hat{\mathbf{n}}$ in expression (15). Without loss of generality we use a coordinate system where the z-axis is aligned with the direction of wave propagation, and the rotation vector $\boldsymbol{\Omega}$ lies in the $-y, z$-plane, as shown in figure 3. In that case, $\hat{\mathbf{s}}_1$ is aligned along the x-axis and $\hat{\mathbf{s}}_2$ along the y-axis.

A perturbation δc_S in velocity corresponds to a perturbation $\delta k/k = -\delta c_S/c_S$ in wavenumber, hence with

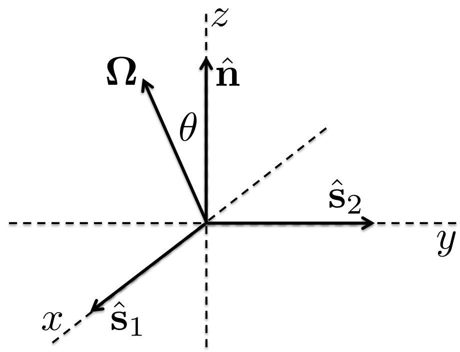

Figure 3. Definition of the unit vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$ for a propagating plane wave.

expression (16)

$$\delta k = -\frac{\Omega \cos\theta}{\omega} k \ . \tag{19}$$

The superposition of the two circularly polarized shear waves is, using expressions (10), (15) and (19) given by

$$\begin{aligned} \mathbf{u} &= \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \exp i \left(k \left(1 - \frac{\Omega\cos\theta}{\omega} \right) - \omega t \right) \\ &+ \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \exp i \left(k \left(1 + \frac{\Omega\cos\theta}{\omega} \right) - \omega t \right) \ . \end{aligned} \tag{20}$$

where we used that $\hat{\mathbf{s}}_1 = \hat{\mathbf{x}}$ and $\hat{\mathbf{s}}_2 = \hat{\mathbf{y}}$. Collecting terms multiplying $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the expression above can be written as

$$\begin{aligned} \mathbf{u} &= \frac{1}{\sqrt{2}} e^{i(kz-\omega t)} \\ &\times \left(\hat{\mathbf{x}} \cos\left(\frac{k\Omega\cos\theta}{\omega} z \right) - \hat{\mathbf{y}} \sin\left(\frac{k\Omega\cos\theta}{\omega} z \right) \right) \ . \end{aligned} \tag{21}$$

This is a linearly polarized wave with polarization vector

$$\hat{\mathbf{q}} = \hat{\mathbf{x}} \cos\left(\frac{k\Omega\cos\theta}{\omega} z \right) - \hat{\mathbf{y}} \sin\left(\frac{k\Omega\cos\theta}{\omega} z \right) \ . \tag{22}$$

The direction of polarization is given by

$$\varphi = \arctan\left(q_y/q_x \right) = -\frac{k\Omega\cos\theta}{\omega} z \ . \tag{23}$$

The time derivative of the location of the wavefronts is given by

$$\dot{z} = c_S = \omega/k \ . \tag{24}$$

This means that rate of change of the orientation of the direction of polarization satisfies $\dot{\varphi} = -(k\Omega\cos\theta/\omega)\dot{z} = -\Omega\cos\theta$. The rate of rotation of the polarization of the S-waves in the transverse plane is thus given by

$$\omega_S = \dot{\varphi} = -\Omega\cos\theta \ . \tag{25}$$

This means that the polarization of the S-wave rotates in the opposite direction as Earth's rotation. The projection of the rotation vector along the direction of propagation is $\Omega\cos\theta$, which is the rotation rate of Foucault's pendulum (Pérez and Pujol, 2015). As shown by equation (25) this rotation is exactly compensated by the rotation of the polarization vector in the plane perpendicular to the direction of wave propagation. The rate of change of the polarization vector follows from expressions (22) and (24) and is given by

$$\dot{\hat{\mathbf{q}}} = -\Omega\cos\theta \ \hat{\mathbf{n}} \times \hat{\mathbf{q}} \ . \tag{26}$$

This amounts to a rotation in the (x, y)-plane with a rotation vector $-\Omega\cos\theta\ \hat{\mathbf{n}}$. In other words, the particle motion of shear waves acts like a Foucault pendulum.

5 DISCUSSION

We have shown observationally and heoretically that seismic rays rays are not deflected by the Coriolis force. The physical reason for this is that the medium that carries the waves co-rotates with Earth's rotation. This means that rays that bounce back and forth in the Earth do not behave as a Foucault pendulum. The presence of Earth's rotation causes a weak $O(\Omega/\omega)$ transverse component for P-waves and a similar longitudinal component for S-waves. More importantly, S-waves in a homogeneous rotating medium have two circular polarizations that propagate at slightly different speeds. The S-waves thus behave similar to a Bravais pendulum. This difference in the S-wave polarization corresponds to the splitting of normal modes by Earth's rotation (Backus and Gilbert, 1961). Superposing the two S-waves with circular polarizations leads to a shear wave with linear polarization whose direction of oscillation changes in exactly the same wave as Fouclault's pendulum. The polarization of shear waves can thus, in principle, be used to measure Earth's rotation.

Suppose one follows a shear wave for a full day. According to expression (25) the total rotation of the polarization is equal to $\Phi = \omega_P \times 1\ day = -\Omega\cos\theta \times 1\ day = -2\pi\cos\theta$. This is not equal to a full cycle, except when the wave propagates along the rotation axis ($\theta = 0$). Rotating Earth over a full cycle thus leaves, in general, an imprint on the polarization of shear waves. This is a seismic example of the Berry phase (Berry, 1984).

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APPENDIX A: CALCULATION OF POLARIZATIONS

We write the polarization vector as a sum of the basis vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$:

$$\hat{\mathbf{q}} = a\hat{\mathbf{n}} + b_1\hat{\mathbf{s}}_1 + b_2\hat{\mathbf{s}}_2 \; . \tag{A1}$$

When using this expansion in the Christoffel equation (11) one needs the cross product of the rotation vector with the basis vectors. If follows from expression (12) that

$$\begin{aligned}
\boldsymbol{\Omega}\times\hat{\mathbf{n}} &= -\Omega\sin\theta\,\hat{\mathbf{s}}_1 \; , \\
\boldsymbol{\Omega}\times\hat{\mathbf{s}}_1 &= \Omega\sin\theta\,\hat{\mathbf{n}} + \Omega\cos\theta\,\hat{\mathbf{s}}_2 \; , \\
\boldsymbol{\Omega}\times\hat{\mathbf{s}}_2 &= -\Omega\cos\theta\,\hat{\mathbf{s}}_1 \; .
\end{aligned} \tag{A2}$$

Inserting the expansion (A1) into the Christoffel equation (11), using expressions (A2), and collecting the coefficients multiplying $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$ gives

$$\begin{aligned}
a &= \frac{\lambda+2\mu}{\rho c^2}a - \frac{2i\Omega}{\omega}\sin\theta\; b_1 \; , \\
b_1 &= \frac{\mu}{\rho c^2}b_1 + \frac{2i\Omega}{\omega}\left(\sin\theta\; a + \cos\theta\; b_2\right) \; , \\
b_2 &= \frac{\mu}{\rho c^2}b_2 - \frac{2i\Omega}{\omega}\cos\theta\; b_1 \; .
\end{aligned} \tag{A3}$$

A1 P waves

Since the imprint of the rotation is assumed to be small ($\Omega/\omega \ll 1$) the P-waves has a polarization that is close to longitudinal. This means that $a = 1$ and that b_1 and b_2 are small. Since the polarization vector is a unit vector, it is to first order only perturbed in the transverse direction, therefore a is not perturbed. The velocity is close to the P-wave velocity in an unrotating medium, hence

$$c_P = \sqrt{\frac{\lambda+2\mu}{\rho}} + \delta c_P \; , \tag{A4}$$

with $\delta c_P \ll c_P$. Using a first order Taylor expansion

$$\frac{1}{c_P^2} = \frac{\rho}{\lambda+2\mu}\left(1 - 2\frac{\delta c_P}{c_P}\right) \; , \tag{A5}$$

Inserting this in expression (A3) gives

$$\begin{aligned}
&\frac{2i\Omega}{\omega}\sin\theta\; b_1 = -2\frac{\delta c_P}{c_P} \; , \\
&\frac{\lambda+\mu}{\lambda+2\mu}b_1 - -\frac{2i\Omega}{\omega}\cos\theta\; b_2 = +\frac{2i\Omega}{\omega}\sin\;\theta \; , \\
&\frac{2i\Omega}{\omega}\cos\theta\; b_1 + \frac{\lambda+\mu}{\lambda+2\mu}b_2 = 0 \; .
\end{aligned} \tag{A6}$$

Since b_1 and b_2 are small, we ignore the products $(\Omega/\omega)b_1$ and $(\Omega/\omega)b_2$ in the last two expressions, which gives

$$b_1 = \frac{2i\Omega}{\omega}\frac{\lambda+2\mu}{\lambda+\mu}\sin\theta \quad , \quad b_2 = 0 \; . \tag{A7}$$

With $a = 1$ this gives the polarization vector (13) for the P-waves. The velocity follows by inserting b_1 from expression (A7) into the first line of equation (A6)

$$\frac{\delta c_P}{c_P} = 2\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\Omega\sin\theta}{\omega}\right)^2 \; , \tag{A8}$$

which gives expression (14).

A2 S waves

For the S-waves the longitudinal polarization is small, so we use that a is small. The shear velocity is given by

$$c_S = \sqrt{\frac{\mu}{\rho}} + \delta c_S \ . \tag{A9}$$

Using a first order Taylor expansion in the perturbation δc_S

$$\frac{1}{c_S^2} = \frac{\rho}{\mu}\left(1 - 2\frac{\delta c_S}{c_S}\right) \ , \tag{A10}$$

Inserting this in expression (A3) and, ignoring cross terms $(\delta c_S/c_S)a$ and $(\Omega/\omega)a$, gives

$$\begin{aligned} \frac{\lambda+\mu}{\lambda+2\mu}a &= \frac{2i\Omega}{\omega}\sin\theta\ b_1 \ , \\ \frac{\delta c_S}{c_S}b_1 &= +\frac{i\Omega}{\omega}\cos\theta\ b_2 \ , \\ \frac{\delta c_S}{c_S}b_2 &= -\frac{i\Omega}{\omega}\cos\theta\ b_1 \ . \end{aligned} \tag{A11}$$

Inserting the equation of the last line into the middle expression gives $(\delta c_S/c_S)^2 = ((\Omega/\omega)\cos\theta)^2$, or

$$\frac{\delta c_S}{c_S} = \pm\frac{\Omega}{\omega}\cos\theta \ . \tag{A12}$$

For the + sign, equation (A11) predicts that $b_2 = -ib_1$, so the normalized polarization vector in the transverse plane in given by $\hat{\mathbf{q}}_S = (1/\sqrt{2})\,(\hat{\mathbf{s}}_1 - i\hat{\mathbf{s}}_2)$. For the $-$ sign in expression (A12), equation (A11) states that $b_2 = +b_1$, hence the normalized polarization in the transverse plane is given by $\hat{\mathbf{q}}_S = (1/\sqrt{2})\,(\hat{\mathbf{s}}_1 + i\hat{\mathbf{s}}_2)$.

Both transverse polarizations are circular. To this circular polarization we should add the small transverse component. The first line of expression (A11) gives, for the used value $b_1 = 1/\sqrt{2}$, the coefficient a follows from expression (18). Using these results gives the polarizations of the S-waves shown in equation (15).