Connection of scattering principles: a visual and mathematical tour

Filippo Brogini & Roel Snieder
Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401, USA

ABSTRACT
Inverse scattering, Green’s function reconstruction, focusing, imaging, and the optical theorem are subjects usually studied as separate problems in different research areas. We speculate that a physical connection exists between them because the equations that rule these scattering principles have a similar functional form. We first lead the reader through a visual explanation of the relationship between these principles and then present the mathematics that illustrates the link between the governing equations of these principles. Throughout this work, we describe the importance of the interaction between the causal and anti-causal Green’s functions.

Key words: focusing, inverse scattering, Green’s function reconstruction, optical theorem, imaging

1 INTRODUCTION
Inverse scattering, Green’s function reconstruction, focusing, imaging, and the optical theorem are subjects usually studied in different research areas such as seismology (Aki and Richards, 2002), quantum mechanics (Rodberg and Thaler, 1967), optics (Born and Wolf, 1999), non-destructive evaluation of material (Shull, 2002), and medical diagnostics (Epstein, 2003).

Inverse scattering (Chadan and Sabatier, 1989; Gladwell, 1993; Colton and Kress, 1998) is the problem of determining the perturbation of a medium (e.g., of a constant velocity medium) from the field scattered by this perturbation. In other words, one aims to reconstruct the properties of the perturbation (represented by the scatterer in Figure 1) from a set of measured data. Inverse scattering takes into account the nonlinearity of the inverse problem, but it also presents some drawbacks: it is improperly posed from the point of view of numerical computations (Dorren et al., 1994), and it requires data recorded at locations usually not accessible due to practical limitations.

Green’s function reconstruction (Wapenaar et al., 2005) is a technique that allows one to reconstruct the response between two receivers (represented by the two triangles at locations \( \mathbf{R}_A \) and \( \mathbf{R}_B \) in Figure 2) from the cross-correlation of the wavefield measured at these two receivers which are excited by uncorrelated sources surrounding the studied system. In the seismic community, this technique is also known as either the virtual source method (Bakulin and Calvert, 2006) or seismic interferometry (Curtis et al., 2006; Schuster, 2009). The first term refers to the fact that the new response is reconstructed as if one receiver had recorded the response due to a virtual source located at the other receiver position; the second indicates that the recording between...
the two receivers is reconstructed through the “interference” of all the wavefields recorded at the two receivers excited by the surrounding sources.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{scattering_principles.png}
\caption{Green’s function reconstruction allows one to reconstruct the response between two receivers (represented by the two triangles at locations $R_A$ and $R_B$).}
\end{figure}

In this paper, the term focusing (Rose, 2001, 2002b) refers to the technique of finding an incident wave that collapses to a spatial delta function $\delta(x - x_0)$ at the location $x_0$ and at a prescribed time $t_0$ (i.e., the wavefield is focused at $x_0$ at $t_0$) as illustrated in Figure 3. In a one-dimensional medium, we deal with a one-sided problem when observations from only one side of the perturbation are available (e.g., due to the practical consideration that we can only record reflected waves); otherwise, we call it a two-sided problem when we have access to both sides of the medium and account for both reflected and transmitted waves.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{focusing.png}
\caption{Focusing refers to the technique of finding an incident wavefield (represented by the dashed lines) that collapses to a spatial delta function $\delta(x - x_0)$ at the location $x_0$ and at a prescribed time $t_0$.}
\end{figure}

In seismology, the term imaging (Claerbout, 1985; Sava and Hill, 2009) refers to techniques that aim to reconstruct an image of the subsurface (Figure 4). Geologist and geophysicists use these images to study the structure of the interior of the Earth and to locate energy resources such as oil and gas. Migration methods are the most widely used imaging techniques and their accuracy depends on the knowledge of the velocity in the subsurface. Migration methods (Bleistein et al., 2001; Biondi, 2006) involve a single scattering assumption (i.e., the Born approximation) because these methods do not take into account the multiple reflections that the waves experience during their propagation inside the Earth; hence the data needs to be preprocessed in a specific way before such methods can be applied.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{imaging.png}
\caption{Imaging refers to techniques that aim to reconstruct an image of the subsurface.}
\end{figure}

The ordinary form of the optical theorem (Rodberg and Thaler, 1967; Newton, 1976) relates the power distinguished from a plane wave incident on a scatterer to the scattering amplitude in the forward direction of the incident field (Figure 5). The scatterer casts a “shadow” in the forward direction where the intensity of the beam is reduced and the forward amplitude is then reduced by the amount of energy carried by the scattered wave. The generalized optical theorem (Heisenberg, 1943) is an extension of the previous theorem and it deals with the scattering amplitude in all the directions; hence it contains the ordinary form as a special case. This theorem relates the difference of two scattering amplitudes to an inner product of two other scattering amplitudes, the generalized optical theorem provides constraints on the scattering amplitudes in many scattering problems (Marston, 2001; Carney et al., 2004). Since these theorems are an expression of energy conservation, they are valid for any scattering system that does not involve attenuation (i.e., no dissipation of energy).

In this tutorial, we refer to the five subjects discussed above as scattering principles because they are all related, in different ways, to a scattering process. These principles are usually studied as independent problems but they are related in various ways; hence, understanding their connections offers insight into each of the principles and eventually may lead to new applications. This work is motivated by a simple idea: because the equations that rule these scattering principles have a similar functional form (see Table 1), there should be a physical connection that could lead to better comprehension of these principles and to possible applications.

To investigate these potential connections, we follow two different paths to provide maximum clarity and physical insights. We first show the relationship between different scattering principles using figures which lead
Scattering loss $\rightarrow$ Scattered power

\[ \text{Scattered power} \]

\[ \text{Incident wave} \]

\[ \text{Scattering loss} \rightarrow \text{Scattered power} \]

Figure 5. The optical theorem relates the power extinguished from a plane wave incident on a scatterer to the scattering amplitude in the forward direction of the incident field.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Principle & Equation \tabularnewline
\hline
1 Inverse Scattering & \( u - u^* = \int u^* f \) \tabularnewline
2 Green’s function reconstruction & \( G - G^* = \int GG^* \) \tabularnewline
3 Optical theorem & \( f - f^* = \int ff^* \) \tabularnewline
4 Imaging & \( I = \int GG^* \) \tabularnewline
\hline
\end{tabular}
\caption{Principles and their governing equations in a simplified form. \( G \) is the Green’s function, \( f \) is the scattering amplitude, and * indicates complex conjugation.}
\end{table}

Table 1.

the reader toward a visual understanding of the connections between the principles; then we illustrate and derive the mathematics that shows the link between the governing equations of some of these principles.

2 VISUAL TOUR

In this section, we lead the reader through a visual understanding of the connections between different scattering principles. Figure 6 illustrates a scattering experiment in a one-dimensional acoustic medium where an impulsive source is placed at the position \( x = 1.44 \) km in the model shown in Figure 8. The incident wavefield, a spatial delta function, propagates toward the discontinuities in the model, interacts with them, and generates outgoing scattered waves. We use a time-space finite difference code with absorbing boundary condition to simulate the propagation of the one-dimensional waves and to produce the numerical examples shown in this section. The computed wavefield is shown in Figure 6 and it represents the causal Green’s function of the system, \( G^+ \). Causality ensures that the wavefield is non zero only in the region delimited by the first arrival (i.e., the direct waves) shown in Figure 7. The slope of the lines, representing the first arrival, depends on the velocity model of Figure 8.

Figure 6. Scattering experiment with a source located at \( x = 1.44 \) km. The traces are recorded by receivers located along the model (shown in Figure 8) with a spacing of 40 m.

Figure 7. Cartoon illustrating a simplified version of the wavefield shown in Figure 6. The wavefield is non zero only in the gray area and the black lines denote the direct waves.

Due to practical limitations, we usually are not able to place a source inside the medium we want to probe, which raises the following question: Is it possible to create the wavefield illustrated in Figure 6 without having a real source at the position \( x = 1.44 \) km? An initial answer to such a question is given by the Green’s function reconstruction. This technique allows one to reconstruct the wavefield that propagates between a virtual source and other receivers located inside the medium (Wapenaar et al., 2005). We remind the reader that this technique yields a combination of the causal wavefield and its time-reversed version (i.e., anti-causal). This is due to the fact that the reconstructed wavefield is propagating between a receiver and a virtual source. Without a real (physical) source, one must have non-zero incident waves to create waves that emanate from a receiver. In the next section we introduce a mathematical argument to explain the interplay of the causal and anti-causal Green’s functions. The fundamental steps to reconstruct the Green’s function are (Curtis et al., 2006).
1. measure the wavefields \( G^+(x, x_{sl}) \) and \( G^+(x, x_{sr}) \) at a receiver located at \( x \) (\( x \) varies from \(-1 \) km to \( 3 \) km) excited by impulsive sources located at both sides of the perturbation \( x_{sl} \) and \( x_{sr} \) (a total of two sources in 1D) as shown in Figure 9;

2. cross-correlate \( G^+(x, x_{sl}) \) with \( G^+(x_{vs}, x_{sl}) \), where \( x_{vs} = 1.44 \) km and \( vs \) stands for virtual source;

3. cross-correlate \( G^+(x, x_{sr}) \) with \( G^+(x_{vs}, x_{sr}) \);

4. sum the results computed at the two previous points to obtain \( G^+(x, x_{vs}) \);

5. repeat this for a receiver located at a different \( x \).

The causal part of wavefield estimated by the Green’s function reconstruction technique is shown in Figure 10 and it is consistent with the result of the scattering experiment produced with a real source located at \( x = 1.44 \) km, shown in Figure 6.

We thus have two different ways to reconstruct the same wavefield, but often we are not able to access a certain portion of the medium we want to study and hence we can’t place any sources or receivers inside it. We next assume that we only have access to scattering data \( R(t) \) measured on the left side of the perturbation, i.e., the reflected impulse response measured at \( x = 0 \) km due to an impulsive source placed at \( x = 0 \) km. This further limitation raises another question: Can we reconstruct the same wavefield shown in Figure 6 having knowledge only of the scattering data \( R(t) \)? Since there are neither real sources nor receivers inside the perturbation, we speculate that the reconstructed wavefield consists of a causal and an anti-causal part, as shown in Figure 11.

For this one-dimensional problem, the answer to this question is given by Rose (2001, 2002a). He shows that we need a particular incident wave in order to collapse the wavefield to a spatial delta function at the desired location after it interacts with the medium, and that this incident wave consists of a spatial delta function followed by the solution of the Marchenko equation, as illustrated in Figure 12.

The Marchenko integral equation (Lamb, 1980; Chadan and Sabatier, 1989) is a fundamental relation of one-dimensional inverse scattering theory. It is an integral equation that relates the reflected scattering amplitude \( R(t) \) to the incident wavefield \( u(t, t_f) \) which will create a focus in the interior of the medium and ultimately gives the perturbation of the medium. The one-dimensional form of this equation is

\[
0 = R(t + t_f) + u(t, t_f) + \int_{-\infty}^{t_f} R(t + t') u(t', t_f) dt', \quad (1)
\]
Figure 11. Wavefield that focuses at $x = 1.44$ km at $t = 0$ s without a source or a receiver at this location. This wavefield consists of a causal ($t > 0$) and an anti-causal ($t < 0$) part.
where \( t_f \) is a parameter that controls the focusing location. We solve the Marchenko equation, using the iterative process described in detail in Rose (2002a), and construct the particular incident wave that focuses at location \( x = 1.44 \) km, as shown in Figures 6 and 11. After seven steps of the iterative process, we inject the incident wave in the model from the left at \( x = 0 \) km and compute the time-space diagram shown in the top panel of Figure 13: it shows the evolution in time of the wavefield when the incident wave is the particular wave computed with the iterative method. The bottom panel of Figure 13 shows a cross-section of the wavefield at the focusing time \( t = 0 \) s: the wavefield vanishes except at location \( x = 1.44 \) km; hence the wavefield focuses at this location.

We create a focus at a location inside the perturbation without having a source or a receiver at such a location and without any knowledge of the medium properties; we only have access to the reflected impulse response measured on the left side of the perturbation. With an appropriate choice of sources and receivers, this experiment can be done in practice (e.g., in a laboratory).

Figure 13 however does not resemble the wavefield shown in Figures 6 and 10. But if we denote the wavefield in Figure 13 as \( w(x, t) \) and its time-reversed version as \( w(x, -t) \), we obtain the wavefield shown in Figure 11 by adding \( w(x, t) \) and \( w(x, -t) \). With this process, we effectively go from one-sided to two-sided illumination because in Figure 11 waves are incident on the scatterer from both sides for \( t < 0 \) s. Burridge (1980) shows similar diagrams and explains how to combine such diagrams using causality and symmetry properties. The upper cone in Figure 11 corresponds to the causal Green’s function and the lower cone represents the anti-causal Green’s function; the relationship between the two Green’s functions is a key element in the next section, where we introduce the homogeneous Green’s function \( G_h \). Note that the wavefield in Figure 11, with a focus in the interior of the medium, is based on reflected data only. We did not use a source or receiver in the medium, and did not know the medium. All necessary is encoded in the reflected wave.

Figure 14 shows four intermediate iterations of the iterative process of Rose. We see that, after adding \( w(x, t) \) to \( w(x, -t) \), the resulting wavefield is increasingly confined in the area defined by the causal and anti-causal cones (i.e., more energy is enclosed inside the two cones after more iterations).

The extension of the iterative process in two dimensions still needs to be investigated; but we conclude this section with a conjecture illustrated in Figure 15. An incident plane wave created by an array of sources is injected in the subsurface where it is distorted due to the variations of the velocity inside the overburden (i.e., the portion of the subsurface that lies above the scatterer). Since we do not know the characteristics of the wavefield when it interacts with the region of the subsurface that includes the scatterer, it is difficult to reconstruct the properties of the scatterer without knowing the medium. Hence, following the insights gained with the one-dimensional problem, we would like to create a special incident wave that, after interacting with the overburden, collapses to a point in the subsurface creating a buried source, as illustrated in Figure 16. In this case, assuming that the medium around the scatterer is homogeneous, we would know the shape of the wavefield that probes the scatterer and partially remove the effect of the overburden, which would facilitate accurate imaging of the scatterer.
Figure 13. Top: After seven steps of the iterative process described in Rose (2002a), we inject the particular incident wave in the model and compute the time-space diagram. Bottom: cross-section of the wavefield at $t = 0$ s.
Figure 14. Four different iterations of the iterative process described in Rose (2002a), we inject the particular incident wave in the model and compute the time-space diagram by adding $w(x, t)$ and $w(x, -t)$. The wavefield is increasingly confined in the area defined by the causal and anti-causal cones (i.e., more energy is enclosed inside these cones after more iterations).
3 REVIEW OF SCATTERING THEORY

We review the theory for the scattering of acoustic waves in a one dimensional medium (also called line) with a constant density, where the scatterer is represented by a perturbation of a constant velocity profile. Here, we introduce the wave equation and Green’s functions that are used in the next section of the paper. Figure 17 shows the geometry of the scattering problem. The perturbation \( c_s(x) \) is superposed on a constant velocity profile \( c_0 \). The following theory is developed in the frequency domain because it simplifies the derivations (e.g., convolution becomes a multiplication and derivatives become multiplications by \( -i\omega \)). We also show the time domain version of some of the following equations because they are more intuitive and allow us to understand the important role played by time-reversal. The Fourier transform convention is defined by \( \hat{f}(\omega) = \int f(t) \exp(-i\omega t) dt \) and \( f(t) = \int \hat{f}(\omega) \exp(i\omega t) d\omega \). Throughout this work, when we deal with a one-dimensional problem, the direction of propagation \( n \) assumes only two values, 1 and \(-1\), which correspond to waves propagating to the right and to the left (Figure 17), respectively.

The equation that governs the motion of the waves in an unperturbed medium with constant velocity \( c_0 \) is the constant-density acoustic homogeneous wave equation

\[
L_0(x, \omega)u_0(n, x, \omega) = 0, \quad (2)
\]

where \( u_0 \) is the displacement wavefield propagating in the \( n \) direction and the differential operator is

\[
L_0(x, \omega) = \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c_0^2}. \quad (3)
\]

The solution of equation 2 is \( u_0(n, x, \omega) = \exp(inx\omega/c_0) \) and its time-domain version is \( \hat{u}_0(n, x, t) = \delta(t - nx/c_0) \), which is a delta function that propagates with velocity \( c_0 \) in the direction of \( n \) representing a physical solution to the wave equation.

The unperturbed Green’s function satisfies the equation

\[
L_0(x, \omega)G^0_0(x, x', \omega) = -\delta(x - x'), \quad (4)
\]

and, in the acoustic one-dimensional case, its frequency-domain expression is (Snyder, 2004):

\[
G^\pm_0(x, x', \omega) = \pm \frac{i}{2k} \exp(\pm ik|x-x'|),
\]

where \( k \equiv \omega/c_0 \). The + and − superscripts of the Green’s function represent the causal and anti-causal Green’s function with outgoing or ingoing boundary conditions (Oristaglio, 1989). In the time domain, causality implies that

\[
G^+_0(x, x', t) = 0 \quad \pm t < |x - x'|/c_0. \quad (6)
\]

Physically, the time-domain Green’s function \( G^+_0(x, x', t) \) represents the displacement at a point \( x \) at time \( t \) due to a point source of unit amplitude applied at \( x' \) at time \( t = 0 \).

Next, we consider the interaction of the wavefield \( u_0 \) with the perturbation \( c_s(x) \) (see Figure 17). This interaction produces a scattered wavefield \( u^s_0 \); hence, the total wavefield can be represented as \( u^\pm = u_0 + u^s_0 \). The + and − superscripts in the total wavefield indicate an initial and a final condition of the wavefield in the time domain, respectively:

\[
\hat{u}^\pm(n, x, t) \rightarrow \hat{u}_0(n, x, t) \quad t \rightarrow \mp \infty. \quad (7)
\]

Physically, condition 7 with plus sign means that the wavefield \( u^+ \), at early times, corresponds to the initial wavefield \( u_0 \) propagating forward in time in the \( n \) direction. The causal and anti-causal wavefields \( u^+ \) and \( u^- \) are related by time-reversal; in fact, each is the time-reversed version of the other \( u^-(t) = u^+(\tau) \). In the frequency domain, time-reversal corresponds to complex conjugation: \( u^-(\omega) = u^+(\omega)^* \). Their time-reversal relationship is better understood by comparing Figures 18a and 18b, which are valid for the velocity model of Figure 8. Figure 18b is obtained by reversing the time axis of Figure 18a. We produced both figures using the same velocity model we used in the Visual Tour section of this paper. In Figure 18a, the initial wavefield is a narrow

\[
\begin{align*}
\text{Incident wave} & \quad \text{Overburden} \\
\text{Focusing} & \quad \text{Scatterer}
\end{align*}
\]

Figure 16. Focusing of the wavefield “at depth”. A special incident wave, after interacting with the overburden, collapses to a point in the subsurface creating a buried source.

\[
\begin{align*}
\text{Incident wave} & \quad \text{Overburden} \\
\text{Focusing} & \quad \text{Scatterer}
\end{align*}
\]

Figure 17. Geometry of the problem for the scattering of acoustic waves in a one dimensional medium with constant density. \( x_A, x_B, x_l \) and \( x_r \) are the coordinates of the receivers (represented by the triangles) and the left and right bounds of our domain, respectively. The perturbation \( c_s \) is superposed on a constant velocity profile \( c_0 \).
Gaussian impulse; while, in Figure 18b, the initial wavefield corresponds to the wavefield at $t = 6$ s in Figure 18a and it coalesces to an outgoing Gaussian pulse.

The total wavefield $u^\pm$ satisfies the wave equation

$$L(x, \omega)u^\pm(n, x, \omega) = 0,$$

(8)

where the differential operator is

$$L(x, \omega) \equiv \left[ \frac{d^2}{dx^2} + \frac{\omega^2}{c(x)^2} \right].$$

(9)

The velocity varies with position, $c(x) = c_0 + c_s(x)$, as illustrated in Figure 17. Using the operator $L$, we define the perturbed Green’s function as the function that satisfies the equation

$$L(x, \omega)G^\pm(x, x', \omega) = -\delta(x - x'),$$

(10)

with the same boundary conditions as equation 5. The Green’s function $G^\pm$ takes into account all the interactions with the perturbation and hence it corresponds to the full wavefield propagating between the points $x'$ and $x$, due to an impulsive source at $x'$.

4 MATHEMATICAL TOUR

In this part of the paper, we lead the reader through a mathematical tour and show that the different scattering principles have a common starting point, i.e., a fundamental equation that reveals the connections between them:

$$G^+(x_A, x_B) - G^-(x_A, x_B) = \sum_{x' = x_1, x_r} m \left[ G^-(x', x_A) \frac{d}{dx}G^+(x, x_B) |_{x=x'} - G^+(x', x_B) \frac{d}{dx}G^-(x, x_A) |_{x=x'} \right]$$

(11)

with

$$m = \begin{cases} -1 & \text{if } x' = x_l \\ +1 & \text{if } x' = x_r \end{cases},$$

where $x_A, x_B, x_l$ and $x_r$ (see Figure 17) are the coordinates of the receivers located at $x_A$ and $x_B$ and the left and right bounds of our domain, respectively. Equation 11 (derived in appendix A) shows a relation between the causal and anti-causal Green’s function and we refer to this expression, throughout the paper, as the representation theorem for the homogenous Green’s function $G_h \equiv G^+ - G^-$ (Oristaglio, 1989), which satisfies the wave equation 10 when its source term is set equal to zero. $G^+$ and $G^-$ both satisfy the same wave equation 10 because the differential operator $L$ is invariant to time-reversal ($L G^+ = -\delta$ and $L G^- = -\delta$); hence, their difference is source free:

$$L(G^+ - G^-) = 0.$$  

(12)

The fact that $G^+ - G^-$ satisfies a homogeneous equation suggests that a combination of the causal and anti-causal Green’s functions is needed to focus the wavefield at a location where there is no real source. This fact has been illustrated in the previous section when we reconstructed the same wavefield using the Green’s function reconstruction technique and the inverse scattering theory (see Figures 10 and 11); in both cases we obtained a combination of the two Green’s functions.

For the remainder of this paper, to be consistent with Budreck and Rose (1990, 1991, 1992), we use the superscripts $+$ and $-$ to indicate the causal and anti-causal behavior of wavefields and Green’s functions; and, for brevity, we omit the dependence on the angular frequency $\omega$.

4.1 Newton-Marchenko equation and generalized optical theorem

In this section, we show that equation 11 is the starting point to derive a Newton-Marchenko equation and a generalized optical theorem. In other words, we demonstrate how lines 1 and 3 of Table 1 are linked to $G_h$. The Newton-Marchenko equation differs from the Marchenko equation because it requires both reflected and transmitted waves as data (Newton, 1980). In contrast to the Marchenko equation, the Newton-Marchenko equation can be extended to two
In a more compact form this can be written as

\[ u^\pm(n, x) = u_0(n, x) + \int dx' G_0^\pm(x, x')L'(x')u^\pm(n, x'), \]

\[ G^\pm(n, x, x') = G_0^\pm(n, x, x') + \int dx''G^\pm(x, x'')L''(x'', x')G_0^\pm(n, x'', x'), \]

\[ f(n, n') = -\int dx'e^{-nikx'}L'(x')u^+(n', x'), \]

where \( L'(x) \equiv L(x) - L_0(x) \) describes the influences of the scatterer (perturbation). Equations 13, 14, and 15 are three different Lippmann-Schwinger equations (Rodberg and Thaler, 1967); they are a reformulation of the scattering problem using linear integral equations with a Green's function kernel. The integral approach is also well suited for the study of inverse problems (Colton and Kress, 1998). Equation 16 is the scattering amplitude (Rodberg and Thaler, 1967) for an incident wave traveling in the \( n \) direction and that is scattered in the \( n' \) direction. We insert equation 15 into 11, simplify considering the fact that \( x_r > x_A, x'_r, x_B \) and \( x_l < x_A, x'_l, x_B \), and using expression 14:

\[ G^+(x_A, x_B) - G^-(x_B, x_A) = \left( \frac{i}{2k} \right) \left[ iku^- (+1, x_A)u^+ (-1, x_B) \ight. \]
\[ + iku^- (-1, x_B)u^+ (+1, x_A) \]
\[ + iku^+ (-1, x_A)u^- (+1, x_B) \]
\[ + iku^+ (+1, x_B)u^- (-1, x_A) \].

In a more compact form this can be written as

\[ G^+(x_A, x_B) - G^-(x_B, x_A) = \left( \frac{i}{2k} \right) \sum_{n=-1,1} u^-(n, x_A)u^+(-n, x_B). \]

Equation 18 is the starting point to derive a Newton-Marchenko equation; we show the full derivation in Appendix B and write the final result:

\[ u^+(+1, x_A) - u^- (+1, x_A) \]
\[ = -\frac{i}{2k} \sum_{n=-1,1} u^-(n, x_A)f(n, x_B), \]
\[ x_B > x_A, x'', \]

\[ u^+(-1, x_A) - u^- (-1, x_A) \]
\[ = -\frac{i}{2k} \sum_{n=-1,1} u^-(n, x_A)f(n, -x_B), \]
\[ x_B < x_A, x''. \]

The system of coupled equations 19 and 20 is our representation of the one-dimensional Newton-Marchenko equation; recognizing that \( u^- = u^* \), it corresponds to line 1 of Table 1.

Next, from expression 19 and using equation 13, \( u^-(n, x) = u^++(-n, x) \), and \( u^\pm = u_0 + u^\pm \), we obtain a generalized optical theorem:

\[ \Re f(-n, n) = -\sum_{n'=-1,1} f(-n, n')f^*(n, n'), \]

where \( n \) assumes the value \(-1 \) or \(+1 \), and \( \Re \) indicates the real part (see line 3 of Table 1). The obtained results are exact because in this one-dimensional framework we do not use any far-field approximations.

## 4.2 Green’s function reconstruction and the Optical theorem

Starting from the three-dimensional version of equation 11, Snieder et al. (2008) showed the connection between the generalized optical theorem and the Green’s function reconstruction. Following their three-dimensional formulation, we illustrate the same result for the one-dimensional problem and show the connection between lines 2 and 3 of Table 1. In this part of the paper, we emphasize the physical meaning of this connection and show the mathematical derivation in the Appendix C. The expression for the Green’s function reconstruction is

\[ \frac{i}{2k} \left[ G^+(x_A, x_B) - G^-(x_A, x_B) \right] \]
\[ = \sum_{x'_{x_A}} G^+(x_A, x')G^-(x_B, x'). \]

Unlike the three-dimensional case, we obtain two different results depending on the configuration of the system. In the first case (Figure 19a), the ordinary optical theorem is derived:

\[ \Re f(n, n) = -\sum_{n'=-1,1} |f(n, n')|^2; \]

in the second case (Figure 19b), we obtain a generalized optical theorem

\[ \Re f(-n, n) = -\sum_{n'=-1,1} f(-n, n')f^*(n, n'), \]
where \( n \) assumes the value \(-1\) or \(+1\) and \( \Re \) indicates the real part in both cases.

![Figure 19](image_url)

**Figure 19.** Configurations of the system used to show the connection between the Green’s function reconstruction and the optical theorem. In both cases, the receivers \( x_A \) and \( x_B \) are located outside the scatterer \( c_s \), which is located at \( x = 0 \).

The above expressions of the optical theorem in one dimension agree with the work of Hovakimian (2005) and differ from their three-dimensional counterpart because they contain the real part of the scattering amplitude instead of the imaginary part. The connection between the Green’s function reconstruction and the generalized optical theorem has not only a mathematical proof, but a physical explanation. The cross-correlation between the Green’s function reconstruction and the generalized optical theorem hold, the ordinary and generalized optical theorem hold, the ordinary form of the optical theorem in one dimension agree with the work of Hovakimian (2005) and imaging. Oristaglio (1989) gives an unusual representation of the spatial delta function that is similar to

### 4.3 Green’s function reconstruction and Imaging in 3D

The three-dimensional version of equation 11 is given by:

\[
G^+(r_A, r_B) - G^-(r_A, r_B) = \int \frac{1}{\rho} \left[ G^-(r, r_A) \nabla G^+(r, r_B) \right] - G^+(r, r_B) \nabla G^-(r, r_A) \cdot n \, d^2r.
\]  

Equation 25 is derived from an acoustic reciprocity theorem of the cross-correlation type, the equation of motion, and the constitutive relation (Wapenaar and Fokkema, 2006). A factor of \( \rho^{-1} \) appears in 25 because Wapenaar and Fokkema (2006) define the Green’s function for a wave equation with a forcing term \(-\rho \delta(r - r')\).

![Figure 20](image_url)

**Figure 20.** Geometry of the scattering problem in three spatial dimensions. The domain consists of a volume \( V \), bounded by a surface \( \partial V \). The unit vector normal to \( \partial V \) is represented by \( n \).

If the sources are in the far-field with respect to the receivers and if the surface \( \partial V \) is a sphere with a large radius, we can apply a radiation boundary condition (Bleistein et al., 2001) to the Green’s function:

\[
\nabla G^\pm(r_A, r_B, r) = \pm \i k G^\pm(r_A, r_B, r)n,  
\]  

where \( n \) is a unit vector normal to the surface \( \partial V \), see Figure 20. Applying this boundary condition to equation 25, we obtain a simplified expression in the frequency domain for the Green’s function reconstruction:

\[
G^+(r_A, r_B) - G^-(r_A, r_B) = 2\i \omega \int_{\partial V} \frac{1}{\rho c} G^-(r, r_A) G^+(r, r_B) d^2r.  
\]  

Equation 27 corresponds to line 2 of Table 1. In global and exploration seismology this is the equation for acoustic seismic interferometry or the virtual source method.

We next establish the connection between equation 25 and imaging. Oristaglio (1989) gives an unusual representation of the spatial delta function that is similar to
5 CONCLUSIONS

In the Visual Tour section, we described the connection between different scattering principles, showing that there are three distinct ways to reconstruct the same wavefield. A physical source, the Green’s function reconstruction technique, and inverse scattering theory allow one to create the same wavefield (see Figure 6) originated by an impulsive source placed at a certain location \(x_s\) (\(x = 1.44\) km in our examples). Green’s function reconstruction tells us how to build an estimate of the wavefield without knowing the medium properties, if we have a receiver at the same location \(x_s\) of the real source and sources surrounding the scattering region. Inverse scattering goes beyond and allows us to focus the wavefield inside the medium (at location \(x_s\)) without knowing its properties, using only data recorded at one side of the medium. We showed that the interaction between causal and anti-causal wavefields is a key element to focus the wavefield where there is no real source; in fact \(G_h = G^+ - G^-\), which satisfies the homogeneous wave equation 10, is a superposition of the causal and anti-causal Green’s functions.

We speculate that many of the insights gained in our one-dimensional framework are still valid in higher dimensions. An extension of this work in two or three dimensions would give us the theoretical tools for many useful practical applications. For example, if we knew how to create the three-dimensional version of the incident wavefield shown in Figure 12, we could focus the wavefield to a point in the subsurface to simulate a source at depth and to record data at the surface (Figure 16); these kind of data are of extreme importance for full waveform inversion techniques (Brenders and Pratt, 2007) and subsalt imaging (Sava and Biondi, 2004). Furthermore, we could possibly concentrate the energy of the wavefield inside a hydrocarbon reservoir to fracture the rocks and improve the production of oil and gas (Beresnev and Johnson, 1994).

In the second part of this tutorial, the Mathematical Tour, we demonstrated that the representation theorem for the homogeneous Green’s function \(G_h\), equation 11, constitutes a theoretical framework for various scattering principles. We showed that all the principles and their equations (see Table 1) rely on \(G_h\) as a starting point for their derivation. As mentioned above, the fundamental role played by the combination of the causal and anti-causal Green’s functions has been evident throughout all the Mathematical Tour: it is this combination that allows one to focus the wavefield to a location where neither a real source nor a receiver can be placed.

ACKNOWLEDGMENTS

The authors would like to thank the members of the Center for Wave Phenomena for their constructive comments. This work was supported by the sponsors of the Consortium Project on Seismic Inverse Methods for Complex Structures at the Center for Wave Phenomena.

References


———, 1980, Inverse scattering, i. one dimension: Journal of Mathematical Physics, 21, 493–505.

Oristaglio, M., 1989, An inverse scattering formula that uses all the data: Inverse Problems, 5, 1097–1105.


APPENDIX A: DERIVATION OF EQUATION 11

Here, we derive equation 11, i.e., the representation theorem for the homogeneous Green’s function. We start
with the equations
\[
\frac{d^2}{dx^2} G^+(x, x_B) + \frac{\omega^2}{c(x)^2} G^+(x, x_B) = -\delta(x - x_B) \tag{A1}
\]
and
\[
\frac{d^2}{dx^2} G^-(x, x_A) + \frac{\omega^2}{c(x)^2} G^-(x, x_A) = -\delta(x - x_A), \tag{A2}
\]
where \(x_A\), \(x_B\), and \(x\) indicate a position between \(x_l\) and \(x_r\) in Figure 17. Next, we multiply equation A1 by \(G^-(x, x_A)\), and equation A2 by \(G^+(x, x_B)\); then we subtract the two results and integrate between \(x_l\) and \(x_r\), yielding
\[
G^+(x_A, x_B) - G^-(x_B, x_A) = \int_{x_l}^{x_r} dx \left[ G^-(x, x_A) \frac{d^2}{dx^2} G^+(x, x_B) + G^+(x, x_B) \frac{d^2}{dx^2} G^-(x, x_A) \right]. \tag{A3}
\]
The right-hand side of the last equation is an exact derivative:
\[
\int_{x_l}^{x_r} dx \left[ G^-(x, x_A) \frac{d^2}{dx^2} G^+(x, x_B) + G^+(x, x_B) \frac{d^2}{dx^2} G^-(x, x_A) \right] = \int_{x_l}^{x_r} dx \left[ G^-(x, x_A) \frac{d}{dx} G^+(x, x_B) + G^+(x, x_B) \frac{d}{dx} G^-(x, x_A) \right] \tag{A4}
\]
hence we obtain the expression for \(G_h\):
\[
G^+(x_A, x_B) - G^-(x_B, x_A) = \sum_{x' = x_l, x_r} m \left[ G^-(x', x_A) \frac{d}{dx} G^+(x, x_B) |_{x = x'} + G^+(x', x_B) \frac{d}{dx} G^-(x, x_A) |_{x = x'} \right] \tag{A5}
\]
with
\[
m = \begin{cases} -1 & \text{if } x' = x_l \\ +1 & \text{if } x' = x_r \end{cases}
\]
where we have used the source-receiver reciprocity relation \(G^\pm(x_A, x_B) = G^\mp(x_B, x_A)\) for the acoustic Green’s function (Wapenaar and Fokkema, 2006).

**APPENDIX B: DERIVATION OF THE NEWTON-MARCHENKO EQUATION**

Inserting expression 15 into the left-hand side of equation 18, using the relation \(u^+ = u_0 + u^+_{ik}\) in the right-hand side of 18, and then inserting 13 into the right-hand side,

we get
\[
e^{-ik|x_A-x_B|} + \int dx'' G^+(x_A, x''') L(x''') e^{-ik|x''-x_B|} + e^{-ik|x_A-x_B|} + \int dx'' G^-(x_A, x''') L(x''') e^{-ik|x''-x_B|} = u^-(+n, x_A) e^{-ik_B} + u^+(-n, x_A) e^{ik_B}
\]
\[
+ \frac{i}{2k} u^-(+n, x_A) \int dx'' e^{-ik(x_B-x'')} L(x''') u^+(+n, x'')
\]
\[
+ \frac{i}{2k} u^+(-n, x_A) \int dx'' e^{ik(x_B-x'')} L(x''') u^+(+n, x''). \tag{B1}
\]

In this one-dimensional problem, we need to consider two different cases: 1) \(x_B > x_A, x''\) and 2) \(x_B < x_A, x''\). Without loss of generality we choose \(x_B > x_A, x''\), and hence obtain
\[
e^{+ikx_B} e^{-ikx_A} + \int dx'' G^+(x_A, x''') L(x''') e^{-ikx_B} + e^{-ikx_B} e^{ikx_A} + \int dx'' G^-(x_A, x''') L(x''') e^{ikx_B}
\]
\[
= u^-(-1, x_A) e^{-ikx_B} + u^+(+1, x_A) e^{ikx_B}
\]
\[
- \frac{i}{2k} u^-(+1, x_A) e^{ikx_B} \int dx'' e^{-ikx''} L(x''') u^+(1, x'')
\]
\[
- \frac{i}{2k} u^+(1, x_A) e^{ikx_B} \int dx'' e^{ikx''} L(x''') u^+(+1, x''). \tag{B2}
\]

The terms inside the brackets in the left-hand side correspond to \(u^-(-1, x_A)\) and \(u^+(+1, x_A)\), respectively; while the integrals in the right-hand side correspond to \(f(+1, -1)\) and \(f(+1, +1)\), respectively. We rewrite equation B2 using 13, 14, and the relation \(f(+n, +n') = f(-n', -n)\), to give
\[
u^+(+1, x_A) - u^-(+1, x_A) = - \frac{i}{2k} \sum_{n=-1,1} u^-(n, x_A) f(n, x_B). \tag{B3}
\]
For the second case \(x_B < x_A, x''\), the solution is
\[
u^-(+1, x_A) - u^-(-1, x_A) = - \frac{i}{2k} \sum_{n=-1,1} u^-(n, x_A) f(n, -x_B). \tag{B4}
\]
APPENDIX C: GREEN'S FUNCTION RECONSTRUCTION AND THE OPTICAL THEOREM

In this appendix we derive the mathematics that shows the connection between the Green’s function reconstruction equation and the optical theorem in one dimension. The expression for the Green’s function reconstruction is

$$\frac{i}{2k} \left[ G^+(x_A, x_B) - G^-(x_A, x_B) \right] = \sum_{x' = x_1, x_r} G^+(x_A, x')G^-(x_B, x'),$$  \hspace{1cm} (C1)

and the Green’s function excited by a point source at $x_s$ recorded at $x_r$ is given by

$$G^+(x_r, x_s) = -\frac{i}{2k} e^{ik|x_s-x_r|} - \frac{i}{2k} e^{ik|x_r-x_s|} f(n, n') e^{ik|x_r|},$$  \hspace{1cm} (C2)

where $f(n, n')$ represents the scattering amplitude (Rodberg and Thaler, 1967), and $n'$ and $n$ represent the directions of the incident wave and the scattered wave, respectively. In the expression above, $Td$ represents the wave traveling directly from the source to the receiver, and term $Ts$ corresponds to the scattered wave that reaches the receivers after interacting with the scatterer. Considering the first configuration (Figure 19a), inserting equation C2 into the right-hand side of equation C1 we get

$$\sum_{x' = x_1, x_r} G^+(x_A, x')G^-(x_B, x') =$$

$$-\frac{i}{2k} \left( \frac{i}{2k} e^{ik(x_B-x_A)} - \frac{i}{2k} e^{-ik(x_B+x_A)} f(-1,1) \right) - \frac{i}{2k} \left( \frac{i}{2k} e^{-ik(x_B-x_A)} - \frac{i}{2k} e^{ik(x_B+x_A)} f^*(-1,1) \right)$$

$$- \left( \frac{i}{2k} \right)^2 e^{ik(x_B-x_A)} \left[ f(-1,1) + f^*(-1,-1) + |f(-1,1)|^2 + |f(-1,1)|^2 \right].$$  \hspace{1cm} (C3)

The terms $T1$ and $T2$ correspond to $G^+(x_A, x_B)$ and $-G^-(x_A, x_B)$, respectively, while the term $T3$ represents the unphysical wave previously discussed in the Mathematical tour; hence, equation C3 simplifies to

$$\sum_{x' = x_1, x_r} G^+(x_A, x')G^-(x_B, x')$$

$$= \frac{i}{2k} \left[ G(x_A, x_B) - G^*(x_A, x_B) \right]$$

$$- \left( \frac{i}{2k} \right)^2 e^{ik(x_B-x_A)} \left[ f(-1,1) + f^*(-1,-1) + |f(-1,1)|^2 + |f(-1,1)|^2 \right].$$  \hspace{1cm} (C4)

For the right-end side of equation C4 to be equal to the left-hand side of equation C1, the expression between the square brackets in term $T3$ should vanish:

$$f(-1,1) + f^*(-1,-1) = -|f(-1,1)|^2 + |f(-1,1)|^2.$$  \hspace{1cm} (C5)

Equation C5 is the expression for the one-dimensional optical theorem (Hovakimian, 2005). The second configuration (Figure 19b) gives

$$\sum_{x' = x_1, x_r} G^+(x_A, x')G^-(x_B, x')$$

$$= \frac{i}{2k} \left[ G^+(x_A, x_B) - G^-*(x_A, x_B) \right]$$

$$- \left( \frac{i}{2k} \right)^2 e^{ik(x_B-x_A)} \left[ f(-1,1) + f^*(1,-1) + f(-1,1)f^*(1,1) + f(-1,1)f^*(1,1) \right].$$  \hspace{1cm} (C6)

In this case, term $T4$ corresponds to the generalized optical theorem in one dimension (Hovakimian, 2005).