Lagrangian Green's function extraction, with applications to potential fields, diffusion, and acoustic waves

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ABSTRACT

The extraction of the Green's function from field fluctuations has led to formulations where for a particular application one retrieves either the real part of the Green's function $G + G^*$ or the imaginary part $G - G^*$. We explore the connection between these different formulations for Green's function extraction for general linear scalar systems and derive equations for the extraction for both the real or imaginary part of the Green's function. We show that for systems that are invariant under time reversal, one can either extract $G - G^*$ when the field is excited by sources on the boundary, or $G + G^*$ when sources are present throughout the volume. The freedom to extract either of these functions is important in situations where the imaginary part of the Green's function vanishes. This is the case, for example in static problems where the Green's function is real. We derive the Green's function extraction for potential field problems and for direct current problems in conducting media. For diffusive fields, the new formalism provides the ability to extract the Green's function either from injection sources or from current sources. We show for acoustic waves that $G - G^*$ can be obtained for systems that satisfy a radiation boundary condition, and that $G + G^*$ can be extracted for systems that satisfy homogeneous boundary conditions. The Green's function extraction formulated here corresponds, for acoustic waves, to a Lagrangian formulation rather than the Hamiltonian (energy) principles were used previously.

1 INTRODUCTION

The Green's function extraction from field fluctuations is an area that has recently gone through a spectacular growth (Wapenaar *et al.*, 2008). The technique is ultimately based on the fluctuation dissipation theorem (Callen & Welton, 1951; Weber, 1956; Tatarskii, 1987) formulated decades ago, but applications flourished when ultrasound measurements showed that this technique could be used in practice (Lobkis & Weaver, 2001; Weaver & Lobkis, 2001; Weaver & Lobkis, 2003; Malcolm *et al.*, 2004). The technique found many applications that include ocean acoustics (Roux & Fink, 2003; Roux *et al.*, 2004; Sabra *et al.*, 2005b), crustal seismology (Campillo & Paul, 2003; Sabra *et al.*, 2005a; Roux *et al.*, 2005), exploration seismology (Bakulin & Calvert, 2006; Mehta *et al.*, 2007; Hornby & Yu, 2007; Miyazawa *et al.*, 2008; Schuster, 2009), structural engineering (Snieder & Şafak, 2006; Snieder *et al.*, 2006;

Kohler *et al.*, 2007; Sabra *et al.*, 2008; Todorovska, 2009), and medical diagnostics (Sabra *et al.*, 2007). There are several derivations for the theory of Green's function extraction that hold for a large class of linear scalar and vector systems (Wapenaar *et al.*, 2006; Snieder *et al.*, 2007; Weaver, 2008).

Derivations for Green's function extraction yield the sum or the difference of the causal Green's function and it's time-reversed counterpart. In the frequency domain, this corresponds to retrieving $G + G^*$ or $G - G^*$, where the superscript asterisk denotes the complex conjugate. (A notable exception is a derivation (Vasconcelos & Snieder, 2009) for the extraction of field *perturbations* for acoustic waves, that lead to expressions for the perturbed Green's function G_S rather than $G_S \pm G_S^*$.) In practice, it is not a problem that one extracts the superposition $G \pm G^*$; because of the causal properties of the Green's function one can reconstruct the time-reversed Green's function from the causal Green's func-

tion. There is confusion, though, in whether the sum or difference is obtained. For example, for acoustic waves some derivations give $G + G^*$, e.g. (Wapenaar *et al.*, 2005), while others give $G - G^*$ (Snieder, 2007). Part of this confusion is related to ambiguities in the definition of the Green's function. Suppose one derivation for Green's function extraction gives an expression for $G - G^*$. Now suppose one defines a new Green's function $\tilde{G}$ that is, in the frequency domain, related to the original Green's function G by the relation $\tilde{G} = -i\omega G$. This corresponds, in the time domain, to considering the time-derivative of the original Green's function. The original expression for $G - G^*$ now leads to a new expression for $(-i\omega)^{-1}(\tilde{G} + \tilde{G}^*)$. Similarly, when one includes a factor i in the excitation of the Green's function (Wapenaar *et al.*, 2005), one obtains a different superposition of G and G^* than when such factor i is excluded (Snieder, 2007). These differences in the definition of the Green's function depend purely on convention, and have no physical meaning.

There is however, a more fundamental reason for analyzing Green's function extraction for $G + G^*$ and $G - G^*$. For wave systems, one can extract the Green's function either from fluctuations excited by sources on a bounding surface (Wapenaar *et al.*, 2005), or from sources throughout the volume (Weaver & Lobkis, 2003). We show in this work that this difference is related to the extraction of different superpositions of G and G^*. The different formulations for Green's function extraction are thus applicable to different distributions for the sources of field fluctuations. One application for which the Green's function extraction has not been applied is to potential field problems. For these problems the Green's function is real, hence $G - G^* = 0$, and therefore the Green's function extraction must be based on the sum $G + G^*$. This application was the original motivation for this work.

In section 2 we analyze general scalar linear systems and generalize an earlier derivation for $G - G^*$ (Snieder *et al.*, 2007) for the extraction of the sum $G + G^*$. The combination of these results makes it possible to extract either G or G^*, hence it is possible to obtain either the causal or the time-reversed Green's function. We apply this general formalism in section 3 to systems that are invariant under time-reversal and show that for such systems $G - G^*$ can be obtained from sources of the field fluctuations on a surface that encloses the receivers, while $G + G^*$ follows from field fluctuations excited by volume sources. In section 4 we apply the general theory to electrostatics or direct current measurement in conducting media. We show that the static Green's function for such problems can be obtained from quasi-static field fluctuations excited by slowly varying electric dipoles throughout the volume. In appendix A we illustrate the theory by considering the special case of a homogeneous medium. We next apply the general theory to the diffusion equation and show that $G - G^*$

can be extracted from injection sources throughout the volume, while $G + G^*$ follows from current sources in the volume. In section 6 we consider acoustic waves and show that $G - G^*$ can be extracted for radiating systems that are excited by sources on a closed surface surrounding receivers, while $G + G^*$ can be extracted from a distribution of volume sources. We verify the expression for $G + G^*$ in appendix B for closed systems using a normal mode expansion.

2   ONE-SIDED GREEN'S FUNCTION EXTRACTION

In this work we consider fields that satisfy the following scalar equation

$$\left(a_n(\mathbf{r}, t) * \frac{\partial^n}{\partial t^n} + \cdots + a_1(\mathbf{r}, t) * \frac{\partial}{\partial t}\right) * u(\mathbf{r}, t)$$
$$= H(\mathbf{r}, t) * u(\mathbf{r}, t) + q(\mathbf{r}, t) . \tag{1}$$

In this expression the asterisk denotes temporal convolution, $H(\mathbf{r}, t)$ is a spatial differential operator, and $q(\mathbf{r}, t)$ is the source of the field $u(\mathbf{r}, t)$. Note that in the important case of Schrödinger's equation, $a_1 = \hbar/i$, with $\hbar$ Planck's constant divided by 2π. This example shows that the coefficients a_n may be complex.

Using the Fourier convention $f(t) = \int f(\omega) \exp(-i\omega t) d\omega$, expression (1) corresponds, in the frequency domain, to

$$\sum_n a_n(\mathbf{r}, \omega)(-i\omega)^n u(\mathbf{r}, \omega) = H(\mathbf{r}, \omega)u(\mathbf{r}, \omega) + q(\mathbf{r}, \omega) . \tag{2}$$

The derivation in the remainder of this work is in the frequency domain, and for brevity we suppress this frequency-dependence in all equations. We consider two states A and B of the field, that are denoted by u_A and u_B, and that are excited by sources q_A and q_B, respectively. An expression for G^* follows by taking expression (2) for state A, multiplying this with u_B^*, and integrating over volume V with boundary ∂V:

$$\sum_n (-i\omega)^n \int_V a_n u_A u_B^* dV$$
$$= \int_V u_B^* H u_A dV + \int_V u_B^* q_A dV . \tag{3}$$

This volume V may be the whole volume over which the field is present, but it may also be an arbitrary sub-volume (Snieder *et al.*, 2007). The multiplication with u_B^* corresponds, in the time domain, to cross-correlation, hence the left hand side expression (3) accounts for correlations of the fields u_A and u_B. For a point sources $q_A(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_A)$ at location $\mathbf{r}_A$, the corresponding field is given by the Green's function: $u_A(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_A)$. A similar notation is used for a point source at $\mathbf{r}_B$ for state B. For these excitations, expression (3) reduces to

$$G^*(\mathbf{r}_A, \mathbf{r}_B) = \sum_n (-i\omega)^n \int_V a_n(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV - \int_V G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) dV . \tag{4}$$

Interchanging the indices A and B, taking the complex conjugate, and using reciprocity ($G(\mathbf{r}_A, \mathbf{r}_B) = G(\mathbf{r}_B, \mathbf{r}_A)$) gives

$$G(\mathbf{r}_A, \mathbf{r}_B) = \sum_n (+i\omega)^n \int_V a_n^*(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV - \int_V G(\mathbf{r}, \mathbf{r}_A) H^*(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B) dV . \tag{5}$$

Expressions (4) and (5) hold for the Green's function and its complex conjugate separately. For time-dependent problems, these functions are nonzero either for positive or negative time, and have for this reason been called *one-sided Green's functions* (Vasconcelos & Snieder, 2009).

   Equations (4) and (5), in their current form, are not directly applicable to the extraction of the Green's function from field fluctuations. In order to extract the Green's function from field fluctuations, one needs to multiply the right hand side of these expressions with uncorrelated noise and rewrite the Green's function multiplied by the noise as the field fluctuations, e.g. (Wapenaar *et al.*, 2005; Snieder *et al.*, 2007). The volume integral $\int G H G^* dV$ is, in general, not symmetric in G and G^* because of the action of the differential operator H. In order to use expressions (4) and (5) for Green's function extraction, one must first rewrite the volume integral in such a way that it is symmetric in G and G^*.

   To illustrate this last point we consider the important case

$$H(\mathbf{r}) f(\mathbf{r}) = \nabla \cdot (D(\mathbf{r}) \cdot \nabla f(\mathbf{r})) . \tag{6}$$

For the diffusion equation D is the diffusion parameter, for acoustic waves $D(\mathbf{r}) = 1/\rho(\mathbf{r})$, with ρ the mass-density, and for Schrödinger's equation $D = -\hbar^2/2m$. For the operator in expression (6) the application of Gauss' law gives

$$\int_V G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) dV = \oint_{\partial V} D(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B) \frac{\partial G(\mathbf{r}, \mathbf{r}_A)}{\partial n} dS - \int_V D(\mathbf{r}) \nabla G^*(\mathbf{r}, \mathbf{r}_B) \cdot \nabla G(\mathbf{r}, \mathbf{r}_A) dV , \tag{7}$$

where $\partial/\partial n$ denotes the derivative normal to the boundary ∂V. The surface integral vanishes when either the field or its normal derivative vanishes on ∂V, in which case

$$\int_V G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) dV = - \int_V D(\mathbf{r}) \nabla G^*(\mathbf{r}, \mathbf{r}_B) \cdot \nabla G(\mathbf{r}, \mathbf{r}_A) dV \tag{8}$$

In the left hand side the operator H acts only on $G(\mathbf{r}, \mathbf{r}_A)$, hence $G^*(\mathbf{r}, \mathbf{r}_B)$ and $G(\mathbf{r}, \mathbf{r}_A)$ enter the left hand side in different ways. Expressions for Green's function extraction are based on the correlation of fluctuations $u(\mathbf{r}_A) = \int G(\mathbf{r}, \mathbf{r}_A) q(\mathbf{r}) dV$ and $u(\mathbf{r}_B) = \int G(\mathbf{r}, \mathbf{r}_B) q(\mathbf{r}) dV$, or on similar expressions containing the gradient of G (see section 4). For this reason, $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$ must enter the resulting integral in the same way, and hence the used expression must be symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$. This is not the case with the left hand side of expression (8), but the right hand side *does* have the required symmetry in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$. In sections 4-6 we apply this property to a number of examples.

   We next analyze the linear combination $G \pm G^*$. Subtracting expression (4) from (5) gives

$$
\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \quad & -2 \sum_{n \ odd} (-i\omega)^n \int Re(a_n(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\
& -2i \sum_{n \ even} (-i\omega)^n \int Im(a_n(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\
& + \int (G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) - G(\mathbf{r}, \mathbf{r}_A) H^*(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B)) dV ,
\end{aligned}
\tag{9}
$$

where Re and Im denote the real and imaginary parts, respectively. Adding expressions (4) and (5) gives

$$
\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \quad & 2i \sum_{n \ odd} (-i\omega)^n \int Im(a_n(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\
& +2 \sum_{n \ even} (-i\omega)^n \int Re(a_n(\mathbf{r})) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\
& - \int (G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) + G(\mathbf{r}, \mathbf{r}_A) H^*(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B)) dV ,
\end{aligned}
\tag{10}
$$

Equation (9) and its application to the diffusion equation, acoustic waves, and quantum mechanics, was discussed earlier (Snieder *et al.*, 2007). In the following we discuss the connection between expressions (9) and (10) and the application to Green's function extraction from field fluctuations.

3   SYSTEMS THAT ARE INVARIANT UNDER TIME-REVERSAL

A connection has been established between time reversal and the Green's function extraction of undamped acoustic waves (Derode *et al.*, 2003). Because of the importance of time-reversal invariance to a large number of applications (Fink, 1997), we first analyze systems that are invariant under time-reversal. Time-reversal corresponds, in the frequency domain to complex conjugation. For such systems equation (2) is equal to its complex conjugate, hence

$$H = H^* \ , \ Re(a_n) = 0 \ \text{ for } n \text{ odd} \ , \ Im(a_n) = 0 \ \text{ for } n \text{ even} \ . \tag{11}$$

Using this in expression (9) gives

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \int_V (G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) - G(\mathbf{r}, \mathbf{r}_A) H(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B)) \, dV \ . \tag{12}$$

Following ref. (Snieder *et al.*, 2007), we define

$$\int_V (fHg - gHf) \, dV = \oint_{\partial V} L_-(f,g) dS \ . \tag{13}$$

For example, for the operator defined in equation (6), it follows from Green's theorem that

$$\int_V (fHg - gHf) \, dV = \oint_{\partial V} D \left(f \frac{\partial g}{\partial n} - \frac{\partial f}{\partial n} g \right) dS \ , \tag{14}$$

hence for this case $L_-(f,g) = D(f\partial g/\partial n - \partial g/\partial n f)$. When the operator H is such that expression (13) holds, one finds for systems invariant under time reversal

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\partial V} L_- \left(G^*(\mathbf{r}, \mathbf{r}_B), G(\mathbf{r}, \mathbf{r}_A) \right) dS \ . \tag{15}$$

Since the right hand side contains a surface integral only, it is possible for systems that are invariant under time-reversal and that satisfy expression (13), to extract the Green's function from field fluctuations that are excited by sources acting on a closed surface that surrounds $\mathbf{r}_A$ and $\mathbf{r}_B$.

Let us next consider the combination $G + G^*$ for systems invariant under time reversal. Inserting the conditions (11) into expression (10) gives

$$
\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \ & 2 \sum_n (-i\omega)^n \int a_n(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \\
& - \int (G^*(\mathbf{r}, \mathbf{r}_B) H(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_A) + G(\mathbf{r}, \mathbf{r}_A) H(\mathbf{r}) G^*(\mathbf{r}, \mathbf{r}_B)) \, dV \ ,
\end{aligned}
\tag{16}
$$

where, for odd n, we used the fact that $iIm(a_n) = a_n$ (because $Re(a_n) = 0$), while for even n the imaginary part of a_n vanishes, hence $Re(a_n) = a_n$. The first term in the right hand side is symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$, which can be exploited to use this form for the extraction of the Green's function from field fluctuations generated by sources in the volume V. The terms in the second volume integral can sometimes be made symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$ using integration by parts. For example, for the operator of expression (6):

$$\int_V (fHG + gHf) \, dV = \oint_{\partial V} D \left(f \frac{\partial g}{\partial n} + \frac{\partial f}{\partial n} g \right) dS - 2 \int_V D(\nabla f \cdot \nabla g) dV \ . \tag{17}$$

It follows from expression (17) that for the operator H from expression (6), the sum $G + G^*$ can be written in a way where each term in the right hand side is symmetric in $G(\mathbf{r}, \mathbf{r}_A)$ and $G^*(\mathbf{r}, \mathbf{r}_B)$, but one cannot avoid a volume integral in the right hand side. This means that Green's function extraction applied to $G + G^*$ from field fluctuations requires sources of these fluctuations throughout the volume. This leads to the important conclusion that for systems that are invariant under time-reversal and with H given by equation (6), one can extract $G - G^*$ from field fluctuations excited on the surface ∂V, while for the extraction of $G + G^*$ one needs field fluctuations excited by sources throughout the volume.

4   GREEN'S FUNCTION EXTRACTION IN ELECTROSTATICS

In linear dielectric media, the electric displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by the relation $\mathbf{D} = \varepsilon\mathbf{E}$, with $\varepsilon(\mathbf{r})$ the electrical permittivity (Griffiths, 1999). Using the field equation $(\nabla \cdot \mathbf{D}) = q$, with $q(\mathbf{r})$ the charge density, and the relation $\mathbf{E} = -\nabla u$, with $u(\mathbf{r})$ the electric potential, the following field equation results

$$0 = \nabla \cdot (\varepsilon(\mathbf{r})\nabla u(\mathbf{r})) + q(\mathbf{r}) \ . \tag{18}$$

In the notation of expression (2), all $a_n = 0$, while H is given by expression (6) with D replaced by ε. We make this replacement throughout this section. If instead of the electrostatic problem we consider direct currents in a conducting medium, then the charge density $q(\mathbf{r})$ is replaced by a volume density of charge injection or extraction rate $-\dot{q}(\mathbf{r})$ (the minus sign comes from the historical convention that the loss of charge from the source region constitutes a positive electric current), and the electric permittivity $\varepsilon(\mathbf{r})$ is replaced by the conductivity $\sigma(\mathbf{r})$. This leaves equation (18) intact with different symbols (Stratton, 1941). This results in a formulation for direct current methods in conducting media.

First note that the Green's function solution to expression (18) is real, hence $G - G^* = 0$ and therefore one cannot retrieve G from the difference $G - G^*$. Using expression (10) together with equation (17) does lead to the following nontrivial relation for $G + G^*$:

$$
\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) \quad &= - \oint_{\partial V} \varepsilon(\mathbf{r}) \left(G(\mathbf{r}, \mathbf{r}_A) \frac{G^*(\mathbf{r}, \mathbf{r}_B)}{\partial n} + \frac{G(\mathbf{r}, \mathbf{r}_A)}{\partial n} G^*(\mathbf{r}, \mathbf{r}_B) \right) dS \\
&\quad + 2 \int_V \varepsilon(\mathbf{r}) \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B) \right) dV .
\end{aligned}
\tag{19}
$$

We now assume that either the potential G or the electrical field $\mathbf{E} = -\nabla G$ vanishes at ∂V, or that the boundary is at infinity. In those cases the surface integral vanishes and

$$
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 2 \int_V \varepsilon(\mathbf{r}) \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B) \right) dV .
\tag{20}
$$

In appendix A we verify expression (20) explicitly for the special case of a homogeneous medium.

In order to establish the connection of this equation with the Green's function extraction from field fluctuations we will use the field generated by an electric dipole distribution $\mathbf{p}(\mathbf{r})$ (Griffiths, 1999)

$$
u(\mathbf{r}_0) = \int_V \left(\nabla G(\mathbf{r}_0, \mathbf{r}) \right) \cdot \mathbf{p}(\mathbf{r}) dV .
\tag{21}
$$

We next consider random dipole sources that are spatially and directionally uncorrelated and satisfy

$$
\langle p_i(\mathbf{r}_1) p_j(\mathbf{r}_2) \rangle = |S|^2 \varepsilon(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij} ,
\tag{22}
$$

where $|S|^2$ measures the strength of the dipole sources. For the moment we consider an ensemble of identical electrostatic systems, each with their own excitation by dipoles, and $\langle \cdots \rangle$ denotes the ensemble average. Multiplying expression (20) with $|S|^2$, using the fact that for this problem G is real, and using the summation convention, gives

$$
\begin{aligned}
G(\mathbf{r}_A, \mathbf{r}_B) |S|^2 \quad &= 2 |S|^2 \int_V \varepsilon(\mathbf{r}) \partial_i G(\mathbf{r}, \mathbf{r}_A) \partial_i G^*(\mathbf{r}, \mathbf{r}_B) dV \\
&= 2 \int_V \int_V \varepsilon(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij} \partial_i G(\mathbf{r}_1, \mathbf{r}_A) \partial_j G^*(\mathbf{r}_2, \mathbf{r}_B) dV_1 dV_2 \\
&= 2 \langle \int_V \partial_i G(\mathbf{r}_1, \mathbf{r}_A) p_i(\mathbf{r}_1) dV_1 \int_V \partial_j G^*(\mathbf{r}_2, \mathbf{r}_A) p_j(\mathbf{r}_2) dV_2 \rangle \\
&= 2 \langle u(\mathbf{r}_A) u^*(\mathbf{r}_B) \rangle ,
\end{aligned}
\tag{23}
$$

where the identity $\int f_i(\mathbf{r}) g_i(\mathbf{r}) dV = \int \int f_i(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij} g_j(\mathbf{r}_2) dV_1 dV_2$ has been used in the second identity, expression (22) in the third equality, and expression (21) in the last identity.

This means that the electrostatic Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$ follows from the ensemble average of the correlation of field fluctuations recorded at $\mathbf{r}_A$ and $\mathbf{r}_B$ that are excited by uncorrelated dipole sources. In reality one may not have an ensemble of identical electrostatic systems, but one may have a system where random sources fluctuate with time. When the characteristic time of the temporal variations in these dipole sources is large compared to the time it takes for light to propagate through the system, the response of the system is quasi-static. In that case the ensemble average can be replaced by a temporal average over the field fluctuations. In fact, the approach to replace an ensemble average by an average over time is common in seismology where averaging over multiple non-overlapping time windows is used to extract the dynamic Green's function (Larose *et al.*, 2006; Shapiro *et al.*, 2005; Sabra *et al.*, 2005a). By applying the same principle to quasi-static field fluctuations one can extract the electrostatic Green's function from temporal field fluctuations.

Charges are not created or destroyed in macroscopic amounts in a source-free region. This means that only dipoles or higher order multi-poles can excite field fluctuations. For the expected small moment fluctuations, where the local charge separation is orders of magnitude smaller than that of the measurement scale, the contribution of the dipole moments dominates the potential field, and, the electric potential is given by equation (21). The occurrence of dipole moments in a material can have four basic causes: electronic, ionic, dipolar and space charge polarization (Khesin *et al.*, 1996).

Relaxation times of the first three processes are usually smaller than 1 μs, while for the last process the relaxation time can be as large as 1 s. These time scales are extremely long compared to the propagation time of light through system of the size of a laboratory experiment or geophysical field experiments. Field fluctuations can be generated, for example in natural rocks by electromagnetic radiation in fracturing rocks or in stressed rocks before fracturing. During a stress increase, and release by fracturing, bonds are broken. The larger the number of cut bonds, the larger is the number of excited atoms, and hence the greater becomes the electromagnetic radiation amplitude. These electromagnetic oscillations behave like surface vibrational optical waves, where positive charges move together in a diametrically opposite phase to the negative ones and decay exponentially into the material like Rayleigh waves. The resulting oscillating electric dipole is the source of the electromagnetic radiation. The pulse amplitude decays due to an interaction with bulk phonons and the life time of the measurable electric field varies between several to 100 μs (Bahat *et al.*, 2005). The application to direct current resistivity problems and the connection with the fluctuation-dissipation theorem is discussed elsewhere in more detail (Slob & Snieder, 2009).

5   DIFFERENT TYPE OF NOISE SOURCES AND THE DIFFUSION EQUATION

In this section we compare different strategies for Green's function extraction for the diffusion equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D\nabla u) + q \tag{24}$$

with $D(\mathbf{r})$ the diffusion parameter. In the notation of equation (1), $a_1 = 1$, all other a_n vanish, and H is given by expression (6). For simplicity we consider the situation where either the field u or the current $D\partial u/\partial n$ vanishes on the boundary ∂V. This is, for example, the case when the boundary is at infinity.

Using expression (9) and equation (14) to convert the volume integral into a vanishing surface integral gives the following expression for $G - G^*$

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega \int_V G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV \tag{25}$$

Alternatively, equation (10) can be used to derive the following expression for $G + G^*$

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B)$$
$$= 2 \int_V D(\mathbf{r}) \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B) \right) dV . \tag{26}$$

where equation (17) has been used to convert a volume integral into a vanishing surface integal.

Both expressions can be used to extract the Green's function from field fluctuations. As shown earlier (Snieder, 2006), for uncorrelated sources that satisfy $\langle q(\mathbf{r}_1)q^*(\mathbf{r}_2)\rangle = \delta(\mathbf{r}_1 - \mathbf{r}_2)|S(\omega)|^2$, with $|S(\omega)|^2$ the power spectrum of the sources, the difference $G - G^*$ can be retrieved from the correlation of the field fluctuations

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2i\omega}{|S(\omega)|^2}\langle u(\mathbf{r}_A)u^*(\mathbf{r}_B)\rangle . \tag{27}$$

Using a reasoning similar to the one used in deriving expression (23) one can extract $G + G^*$ by assuming that random spatially and directionally uncorrelated current sources $\mathbf{J}$ are present that satisfy

$$\langle J_i(\mathbf{r}_1)J_j^*(\mathbf{r}_2)\rangle = |S(\omega)|^2 D(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta_{ij} . \tag{28}$$

The current sources generate field fluctuations $u^{(J)}$, and using the same arguments as used in the derivation of expression (23) it follows that

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2}{|S(\omega)|^2}\langle u^{(J)}(\mathbf{r}_A)u^{(J)*}(\mathbf{r}_B)\rangle . \tag{29}$$

Expressions (27) and (29) show that both $G - G^*$ and $G + G^*$ can be extracted from field fluctuations, but for $G - G^*$ these fluctuations must be excited by sources of the field itself (monopole sources), whereas for $G + G^*$ the fluctuations must be excited by current sources (dipole sources). This means that depending on the type of sources that actually is present in the system under consideration, one may be able to extract either $G - G^*$ or $G + G^*$. Of course, one may make linear combinations of expressions (27) and (29) to accommodate any combination of monopole and dipole sources.

6   DIFFERENT TYPE OF NOISE SOURCES AND ACOUSTIC WAVES

As a next example we treat acoustic waves where the pressure p and sources q are related by

$$\frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2} = \nabla \cdot \left(\frac{1}{\rho}\nabla p\right) + q , \tag{30}$$

with ρ the mass density and c the wave velocity. In the notation of expression (1), $a_2 = 1/\rho c^2$, all other a_n vanish, and H is given by equation (6) with D replaced by $1/\rho$. The excitation q is related to the injection sources Q and body forces $\mathbf{f}$ by the relation $q = \dot{Q}/\kappa - \nabla \cdot (\mathbf{f}/\rho)$, where κ^{-1} is the bulk modulus, and the dot denotes the time-derivative. With the employed Fourier convention, this corresponds, in the frequency domain, to

$$q(\mathbf{r}) = -i\omega\frac{Q(\mathbf{r})}{\kappa(\mathbf{r})} - \nabla \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})}\right) . \tag{31}$$

In the following we assume there is no attenuation.

We first take ∂V a large sphere and assume a radiation boundary condition on this sphere

$$\frac{\partial p}{\partial n} = ikp \text{ on } \partial V , \tag{32}$$

with $k = \omega/c$. Using equation (9) with expression (14) gives the following expression for $G - G^*$

$$G(\mathbf{r}_A, \mathbf{r}_B) - G^*(\mathbf{r}_A, \mathbf{r}_B) = 2i\omega \oint_{\partial V} \frac{1}{\rho c} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dS . \tag{33}$$

This well-known equation (Derode *et al.*, 2003; Wapenaar & Fokkema, 2006; Snieder *et al.*, 2007) states that the Green's function can be extracted from field fluctuations excited by sources on a closed surface surrounding $\mathbf{r}_A$ and $\mathbf{r}_B$.

When f and g satisfy the radiation boundary condition (32), $f\partial g^*/\partial n + g^*\partial f/\partial n = 0$. This term also vanishes when either the field or its normal derivative vanishes at the boundary ($p = 0$ or $\partial p/\partial n = 0$ at ∂V). We show in appendix B that for homogeneous boundary conditions the Green's function is real, hence $G - G^* = 0$, and only $G + G^*$ can provide a nontrivial expression for Green's function extraction.

Using the radiation boundary condition or homogeneous boundary condition in expressions (10) and (17) gives the following for the sum $G + G^*$

$$G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 2 \int_V \frac{1}{\rho} \left((\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) - \frac{\omega^2}{c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV . \tag{34}$$

The last term on the right hand side comes from the contribution of a_2 to expression (10). This term has the same dependence on the Green's function as does the surface integral in expression (33), but now appears in a volume integral.

When the field satisfies the radiation boundary condition (32), the difference $G - G^*$ can be extracted by cross-correlating the pressure field excited by uncorrelated sources with power spectrum $|S(\omega)|^2$ at the bounding surface ∂V which leads to an expression identical to equation (27) but now for acoustic waves (Snieder *et al.*, 2009a). The situation is more complicated for the case of equation (34) for the sum $G + G^*$, which is applicable under radiation boundary conditions or under homogeneous boundary conditions.

For a general excitation $q(\mathbf{r})$ the pressure is given by $p(\mathbf{r}_0) = \int G(\mathbf{r}_0, \mathbf{r}) q(\mathbf{r}) dV$. Using the decomposition (31) in injection sources and body forces, the pressure can be written as

$$p(\mathbf{r}_0) = -i\omega \int G(\mathbf{r}_0, \mathbf{r}) \frac{Q(\mathbf{r})}{\kappa(\mathbf{r})} dV - \int G(\mathbf{r}_0, \mathbf{r}) \nabla \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})} \right) dV . \tag{35}$$

Using Gauss's law, the last integral can be written as $\oint_{\partial V} G(\mathbf{r}_0, \mathbf{r}) \mathbf{f}(\mathbf{r})/\rho(\mathbf{r}) \cdot d\mathbf{S} - \int (\nabla G(\mathbf{r}_0, \mathbf{r})) \cdot (\mathbf{f}(\mathbf{r})/\rho(\mathbf{r})) \, dV$. When the body force vanishes on the boundary ∂V, the first term in this expression vanishes, and the pressure is given by

$$p(\mathbf{r}_0) = -i\omega \int G(\mathbf{r}_0, \mathbf{r}) \frac{Q(\mathbf{r})}{\kappa(\mathbf{r})} dV + \int (\nabla G(\mathbf{r}_0, \mathbf{r})) \cdot \left(\frac{\mathbf{f}(\mathbf{r})}{\rho(\mathbf{r})} \right) dV . \tag{36}$$

Consider first an experiment where only random injection sources excite field fluctuations and that these injection sources are spatially uncorrelated and satisfy

$$\langle Q(\mathbf{r}_1) Q^*(\mathbf{r}_2) \rangle = \kappa(\mathbf{r}_1) |S(\omega)|^2 \delta(\mathbf{r}_1 - \mathbf{r}_2) . \tag{37}$$

We assume there are no body forces acting ($\mathbf{f} = 0$), and denote the associated pressure response by p^Q. This response is given by the first term in the right hand side of equation (36), and the cross-correlation of the pressure fluctuations satisfies

$$\begin{aligned} \langle p^Q(\mathbf{r}_A) p^{Q*}(\mathbf{r}_B) \rangle \quad &= \omega^2 \langle \int G(\mathbf{r}_A, \mathbf{r}_1) \frac{Q(\mathbf{r}_1)}{\kappa(\mathbf{r}_1)} dV_1 \int G^*(\mathbf{r}_B, \mathbf{r}_2) \frac{Q(\mathbf{r}_2)}{\kappa(\mathbf{r}_2)} dV_2 \rangle \\[2mm] &= \omega^2 \int \int \frac{G(\mathbf{r}_A, \mathbf{r}_1) G^*(\mathbf{r}_B, \mathbf{r}_2)}{\kappa(\mathbf{r}_1) \kappa(\mathbf{r}_2)} \langle Q(\mathbf{r}_1) Q^*(\mathbf{r}_2) \rangle dV_1 dV_2 \\[2mm] &= |S(\omega)|^2 \int \frac{\omega^2}{\rho c^2} G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dV , \end{aligned} \tag{38}$$

where expression (37) and the relation $\kappa = \rho c^2$ have been used in the last identity. Consider next another experiment where the field fluctuations are excited by uncorrelated body forces that satisfy

$$\langle f_i(\mathbf{r}_1) f_j^*(\mathbf{r}_2) \rangle = \rho(\mathbf{r}_1) |S(\omega)|^2 \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta_{ij} , \tag{39}$$

and that there are no injection sources ($Q = 0$). The pressure field p^f for these sources is given by the last term of expression (36), and the correlation of the field fluctuations satisfies

$$
\begin{aligned}
\langle p^f(\mathbf{r}_A) p^{f*}(\mathbf{r}_B) \rangle \quad &= \langle \int (\partial_i G(\mathbf{r}_A, \mathbf{r}_1)) \frac{f_i(\mathbf{r}_1)}{\rho(\mathbf{r}_1)} dV_1 \int (\partial_j G^*(\mathbf{r}_B, \mathbf{r}_2)) \frac{f_j^*(\mathbf{r}_2)}{\rho(\mathbf{r}_2)} dV_2 \rangle \\[2mm]
&= \int \int \frac{(\partial_i G(\mathbf{r}_A, \mathbf{r}_1))(\partial_j G^*(\mathbf{r}_B, \mathbf{r}_2))}{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)} \langle f_i(\mathbf{r}_1) f_j^*(\mathbf{r}_2) \rangle dV_1 dV_2 \\[2mm]
&= |S(\omega)|^2 \int \frac{1}{\rho} (\nabla G(\mathbf{r}_A, \mathbf{r})) \cdot (\nabla G^*(\mathbf{r}_B, \mathbf{r})) dV \ ,
\end{aligned}
\tag{40}
$$

where expression (39) is used on the last identity. Inserting equations (38) and (40) into equation (34) finally gives

$$
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = \frac{2}{|S(\omega)|^2} \left(\langle p^f(\mathbf{r}_A) p^{f*}(\mathbf{r}_B) \rangle - \langle p^Q(\mathbf{r}_A) p^{Q*}(\mathbf{r}_B) \rangle \right) \ .
\tag{41}
$$

The sum $G + G^*$ can thus be found by measuring the field fluctuations generated by injection sources and body forces and subtracting the cross-correlations of the associated pressure fluctuations. This procedure is usually not practical because one usually does not have the ability to separately record the field fluctuations excited by injection sources and by body forces.

Unfortunately one cannot use expression (34) for the much more practical case where injection sources and body forces act *simultaneously*. Suppose the injection sources and body forces satisfy equations (37) and (39), and that these different types of sources are mutually uncorrelated: $\langle Qf \rangle = 0$. Following the steps of the derivation of equations (38) and (40) then gives the *sum* $\langle p^f(\mathbf{r}_A) p^{f*}(\mathbf{r}_B) \rangle + \langle p^Q(\mathbf{r}_A) p^{Q*}(\mathbf{r}_B) \rangle$ rather than the *difference* that is needed in expression (34). Yet the minus sign in the latter expression is correct, as we verify in appendix B.

Using the equation of motion for acoustic waves ($-i\omega\rho\mathbf{v} = -\nabla p$, with $\mathbf{v}$ the particle velocity), one can write expression (34) as

$$
G(\mathbf{r}_A, \mathbf{r}_B) + G^*(\mathbf{r}_A, \mathbf{r}_B) = 4 \int_V \left(\frac{\rho\omega^2}{2} \mathbf{G}^{(v)}(\mathbf{r}, \mathbf{r}_A) \cdot \mathbf{G}^{(v)*}(\mathbf{r}, \mathbf{r}_B) - \frac{\omega^2}{2\kappa} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV \ ,
\tag{42}
$$

where $\mathbf{G}^{(v)}(\mathbf{r}, \mathbf{r}')$ is the particle velocity associated with the Green's function $G(\mathbf{r}, \mathbf{r}')$. When $\mathbf{r}_A = \mathbf{r}_B$, the integral in the right hand side is given by $\int_V \left((1/2)\rho\omega^2 |\mathbf{G}^{(v)}(\mathbf{r}, \mathbf{r}_A)|^2 - (\omega^2/2\kappa)|G(\mathbf{r}, \mathbf{r}_A)|^2 \right) dV$. This difference of kinetic and potential energy is the Lagrangian of the acoustic field. Previous formulations for Green's function extraction were based on energy principles (Snieder *et al.*, 2007; Snieder *et al.*, 2009b), hence they were described by a Hamiltonian formulation. As shown in equation (42), the approach taken in this paper is, for acoustic waves, based on the Lagrangian. For this reason we use the phrase *Lagrangian Green's function extraction*.

7 DISCUSSION

Traditionally the theory of Green's function extraction from field fluctuations is usually presented as if one can either extract $G - G^*$ or $G + G^*$, depending on the chosen convention for the Green's function. We show that for a large class of linear scalar system one can extract both $G - G^*$ or $G + G^*$ from field fluctuations, but that the requirement for the distribution of the sources of these fluctuations may differ. For systems invariant under time reversal, one needs uncorrelated sources on a closed surface surrounding the receivers to extract $G - G^*$, while one needs sources throughout the volume to extract $G + G^*$. This means that the character of the source distribution (surface sources vs. volume sources) determines whether one retrieves $G - G^*$ or $G + G^*$.

For static systems the Green's function is real, hence $G - G^*$ is uninformative. We show that for electrostatic systems, the Green's function can be extracted from quasi-static field fluctuations excited by uncorrelated electric dipole sources. This new result may have applications in geophysics where random dipoles can be present due to the charge separation that occurs in fracture processes, streaming potentials, and the breakup of meniscuses during drainage of porous media.

Just as for systems that are invariant under time-reversal, one needs different sources of field fluctuations depending on whether one extracts $G - G^*$ or $G + G^*$ for the diffusion equation. For fields satisfying this equation one needs volume sources in both cases, but injection sources (monopoles) allow for the retrieval of $G - G^*$ while current sources (dipoles) make it possible to extract $G + G^*$. An example of a diffusive system with monopole sources is a fluid in which chemicals released by moving micro-organisms spread diffusively. A porous medium where fluid is injected randomly is an example of a diffusive system with injection sources.

We have shown that for acoustic waves that satisfy a radiation boundary condition at a closed surface, uncorrelated sources at that surface suffice for the extraction of $G - G^*$. When volume sources are present, one can obtain the Green's function from the sum $G + G^*$. A practical limitation of the latter approach is that in this case one needs to record the field fluctuations generated by injection sources and by body forces separately. The ability to extract $G - G^*$ and $G + G^*$ from field fluctuations gives the principle of Green's function extraction a new degree of freedom that has the potential to open up new opportunities for the extraction of the system response from field fluctuations.

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APPENDIX A: APPLICATION OF EQUATION (20) TO A HOMOGENEOUS MEDIUM.

For a homogenous medium with electric permittivity ε the Green's function is given by

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi\varepsilon|\mathbf{r}_1 - \mathbf{r}_2|} \ . \tag{A1}$$

Inserting this into equation (20) gives

$$\int_V \varepsilon \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)\right) dV$$
$$= \frac{1}{16\pi^2\varepsilon} \int_V \nabla \frac{1}{|\mathbf{r}-\mathbf{r}_A|} \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}_B|} dV \ . \tag{A2}$$

The **r**-derivatives can be replaced by derivatives acting on $\mathbf{r}_A$ and $\mathbf{r}_B$ but with opposite signs, giving

$$\int_V \varepsilon \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)\right) dV$$
$$= \frac{\nabla_A \cdot \nabla_B}{16\pi^2\varepsilon} \int_V \frac{1}{|\mathbf{r}-\mathbf{r}_A||\mathbf{r}-\mathbf{r}_B|} dV. \tag{A3}$$

Taking the volume V as a ball with radius $\Delta > |\mathbf{r}_A - \mathbf{r}_B|$ around the point $\mathbf{r}_A$ and switching to spherical coordinates yields

$$\int_V \varepsilon \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)\right) dV$$
$$= \frac{\nabla_A \cdot \nabla_B}{8\pi\varepsilon} \int_{r=0}^{\Delta} \int_{\theta=0}^{\pi} \frac{r\sin(\theta)}{\sqrt{r^2 + r_{AB}^2 + 2rr_{AB}\cos(\theta)}} d\theta dr, \tag{A4}$$

where use has been made of the fact that $|\mathbf{r} - \mathbf{r}_A| = r$, $|\mathbf{r}-\mathbf{r}_B| = \sqrt{r^2 + r_{AB}^2 + 2rr_{AB}\cos(\theta)}$, with $r_{AB} = |\mathbf{r}_A - \mathbf{r}_B|$ and θ the angle between $\mathbf{r} - \mathbf{r}_A$ and $\mathbf{r}_A - \mathbf{r}_B$. Since the integrand in the left hand side of expression (A4) does, for the employed coordinate system, not depend on the polar angle φ, the integration over this angle gives a contribution 2π to the right hand side. The θ integral is elementary and gives

$$\int_V \varepsilon \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)\right) dV$$
$$= -\nabla_A \cdot \nabla_B \frac{1}{8\pi\varepsilon r_{AB}} \int_{r=0}^{\Delta} \left. \sqrt{r^2 + r_{AB}^2 + 2rr_{AB}\cos(\theta)} \right|_{\theta=0}^{\pi} dr, \tag{A5}$$
$$= -\nabla_A \cdot \nabla_B \frac{1}{8\pi\varepsilon r_{AB}} \left(\int_{r=0}^{\Delta} |r - r_{AB}| dr - \int_{r=0}^{\Delta} (r + r_{AB}) dr\right).$$

The first integral can be split in two intervals $0 < r < r_{AB}$ and $r_{AB} < r < \Delta$. The three resulting integrals are given by

$$\int_{r=0}^{r_{AB}} (r_{AB} - r) dr = \frac{r_{AB}^2}{2}, \tag{A6}$$

$$\int_{r=r_{AB}}^{\Delta} (r - r_{AB}) dr = \frac{\Delta^2}{2} - r_{AB}\Delta + \frac{r_{AB}^2}{2}, \tag{A7}$$

$$\int_{r=0}^{\Delta} (r + r_{AB}) dr = \frac{\Delta^2}{2} - r_{AB}\Delta, \tag{A8}$$

from which it is clear that the radius Δ does not contribute to the final result, hence the location of the boundary is irrelevant. This finally leads to the correct Green's function:

$$\int_V \varepsilon \left(\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)\right) dV = -\frac{\nabla_A \cdot \nabla_B}{8\pi\varepsilon} r_{AB}$$
$$= \frac{1}{4\pi\varepsilon r_{AB}} = G(\mathbf{r}_A - \mathbf{r}_B) \ . \tag{A9}$$

APPENDIX B: DERIVING EQUATION (34) FROM A NORMAL MODE EXPANSION

In this appendix we consider the normal mode expansion of the Green's function for acoustic waves. The modes $u_n(\mathbf{r})$ satisfy equation (30) without an excitation

$$\nabla \cdot \left(\frac{1}{\rho}\nabla u_n\right) + \frac{\omega_n^2}{\kappa} u_n = 0 \ , \tag{B1}$$

In the absence of attenuation the modes are real. The orthogonality relation of the modes follow from a standard derivation consisting of multiplying this expression with a mode u_m, integrating over volume, interchanging n and m and subtracting, applying Gauss' theorem and using that on the boundary either $u = 0$ or $\partial u / \partial n = 0$. This gives the following orthogonality relation that also defines the normalization of the modes

$$\int \frac{1}{\kappa} u_n u_m dV = \delta_{nm} \ . \tag{B2}$$

Eliminating u_n/κ in expression (B2) using the second term of equation (B1) and applying Gauss' theorem gives the following alternative orthogonality relation

$$\int \frac{1}{\rho} (\nabla u_n \cdot \nabla u_m) dV = \omega_n^2 \delta_{nm} \ . \tag{B3}$$

The Green's function has the following normal mode expansion (Snieder, 2004)

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{u_n(\mathbf{r}_1) u_n(\mathbf{r}_2)}{\omega_n^2 - \omega^2} \ . \tag{B4}$$

Since the modes are real, G is a real function as well.

We next express the integrals in the right hand side of equation (34) in normal modes. Using the previous expression and the relation $\kappa = \rho c^2$ gives

$$\int_V \frac{\omega^2}{\rho c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV = \sum_{n,m} \frac{\omega^2}{(\omega_n^2 - \omega^2)(\omega_m^2 - \omega^2)} \left(\int \frac{1}{\kappa(\mathbf{r})} u_n(\mathbf{r}) u_m(\mathbf{r}) dV \right) u_n(\mathbf{r}_A) u_m(\mathbf{r}_B) \ . \tag{B5}$$

The orthogonality relation (B2) reduces the double sum in the right hand side to a single sum:

$$\int_V \frac{\omega^2}{\rho c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) dV = \omega^2 \sum_n \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{(\omega_n^2 - \omega^2)^2} \ . \tag{B6}$$

This expression differs from the the normal mode expansion (B4) by the square of the frequency difference in the denominator.

The first term in the right hand side of expression (34) can also be expressed in the normal mode expansion (B4):

$$\int_V \frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A)) \cdot (\nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV = \sum_{n,m} \frac{\omega^2}{(\omega_n^2 - \omega^2)(\omega_m^2 - \omega^2)} \left(\int \frac{1}{\rho(\mathbf{r})} (\nabla u_n(\mathbf{r}) \cdot \nabla u_m(\mathbf{r})) dV \right) u_n(\mathbf{r}_A) u_m(\mathbf{r}_B) \ . \tag{B7}$$

With the orthogonality relation (B3) the double sum can be reduced to a single sum over modes

$$\int_V \frac{1}{\rho} (\nabla G(\mathbf{r}, \mathbf{r}_A)) \cdot (\nabla G^*(\mathbf{r}, \mathbf{r}_B)) dV = \sum_n \omega_n^2 \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{(\omega_n^2 - \omega^2)^2} \ . \tag{B8}$$

The dominant contribution to the response comes from eigenfrequencies ω_n close to the frequency of excitation (ω). A comparison of expressions (B6) and (B8) shows that the two integrals in equation (34) are of the same order of magnitude.

Subtracting equations (B6) and (B8) gives

$$\int_V \frac{1}{\rho} \left((\nabla G(\mathbf{r}, \mathbf{r}_A) \cdot \nabla G^*(\mathbf{r}, \mathbf{r}_B)) - \frac{\omega^2}{c^2} G(\mathbf{r}, \mathbf{r}_A) G^*(\mathbf{r}, \mathbf{r}_B) \right) dV = \sum_n \frac{u_n(\mathbf{r}_A) u_n(\mathbf{r}_B)}{\omega_n^2 - \omega^2} \ . \tag{B9}$$

By virtue of the expansion (B4) this is equal to the Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$. Since the Green's function is real, expression (B9) constitutes an alternative proof of equation (34). Note that in order to obtain this result it is essential to subtract expressions (B6) and (B8) because it is this subtraction that gives the contribution $(\omega_n^2 - \omega^2)$ needed to make the numerator in expressions (B6) and (B8) equal to that of the Green's function in equation (B4).