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General representation theorem for perturbed media and application to Green's function retrieval for scattering problems

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ABSTRACT
Green's function reconstruction relies on representation theorems. For acoustic waves, it has been shown theoretically and observationally that a representation theorem of the correlation-type leads to the retrieval of the Green's function by cross-correlating fluctuations recorded at two locations and excited by uncorrelated sources. We extend the theory to any system that satisfies a linear partial differential equation, and define an *interferometric operation* that is more general than cross-correlation for the reconstruction. We analyze Green's function reconstruction for perturbed media and establish a representation theorem specifically for field perturbations. That representation is then applied to the general treatment of scattering problems, enabling interpretation of the contributions to Green's function reconstruction in terms of direct and scattered waves. Perhaps surprising, Green's functions that account for scattered waves cannot be reconstructed from scattered waves alone. For acoustic waves, retrieval of scattered waves also requires cross-correlating direct and scattered waves at receiver locations. The addition of cross-correlated scattered waves with themselves is necessary to cancel the spurious events that contaminate the retrieval of scattered waves from the cross-correlation of direct with scattered waves. We illustrate these concepts with numerical examples for the case of an open scattering medium. The same reasoning holds for the retrieval of any type of perturbations, and can be applied to perturbation problems such as electromagnetic waves in conductive media, and elastic waves in heterogeneous media.

Key words: representation theorem – Green's function retrieval – interferometry – perturbation theory – scattering problem.

1 INTRODUCTION

The extraction of Green's functions from wave field fluctuations has recently received considerable attention. The technique, known in much of the literature as interferometry, is described in tutorials (Curtis *et al.* 2006; Larose *et al.* 2006; Wapenaar *et al.* 2007) and has been applied to a large variety of fields including ultrasonics (Lobkis & Weaver 2001; Malcolm *et al.* 2004; Weaver & Lobkis 2001), global (Campillo & Paul 2003; Sabra *et al.* 2005a; Shapiro *et al.* 2005) and exploration (Bakulin & Calvert 2006; Miyazawa *et al.* 2008) seismology, helioseismology (Rickett & Claerbout 1999), medical imaging (Sabra *et al.* 2007), structural engineering (Kohler *et al.* 2007; Snieder & Safak 2006; Thompson & Snieder 2006), and ocean acoustics (Roux & Kuperman 2004; Sabra *et al.* 2005b). The theory relies on representation theorems (of either the convolution or correlation type) and allows for the retrieval of Green's functions for acoustic (Wapenaar & Fokkema 2006), elastic (Snieder 2002; Van Manen *et al.* 2006; Wapenaar *et al.* 2004), and electromagnetic (Slob *et al.* 2007; Wapenaar *et al.* 2006) waves. For acoustic media, the impulse response between two receivers is retrieved by cross-correlating and summing the signals recorded by the two receivers

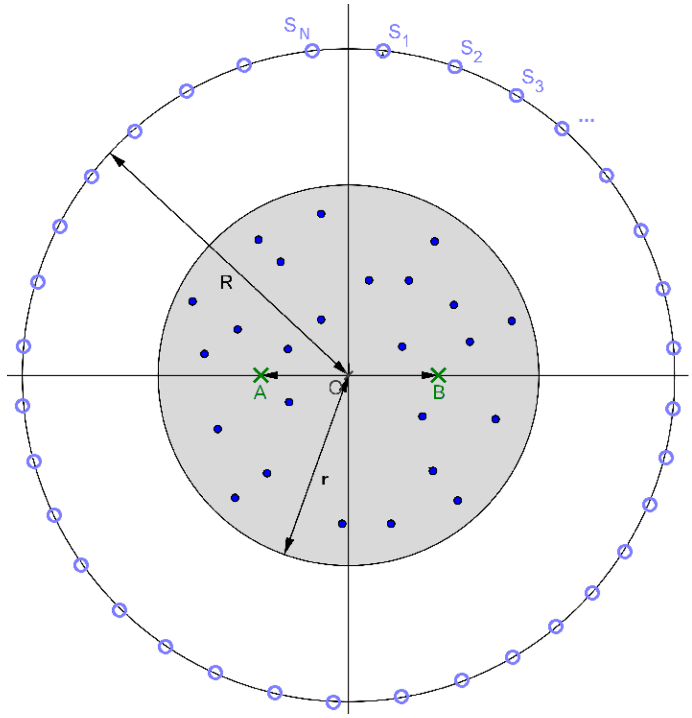
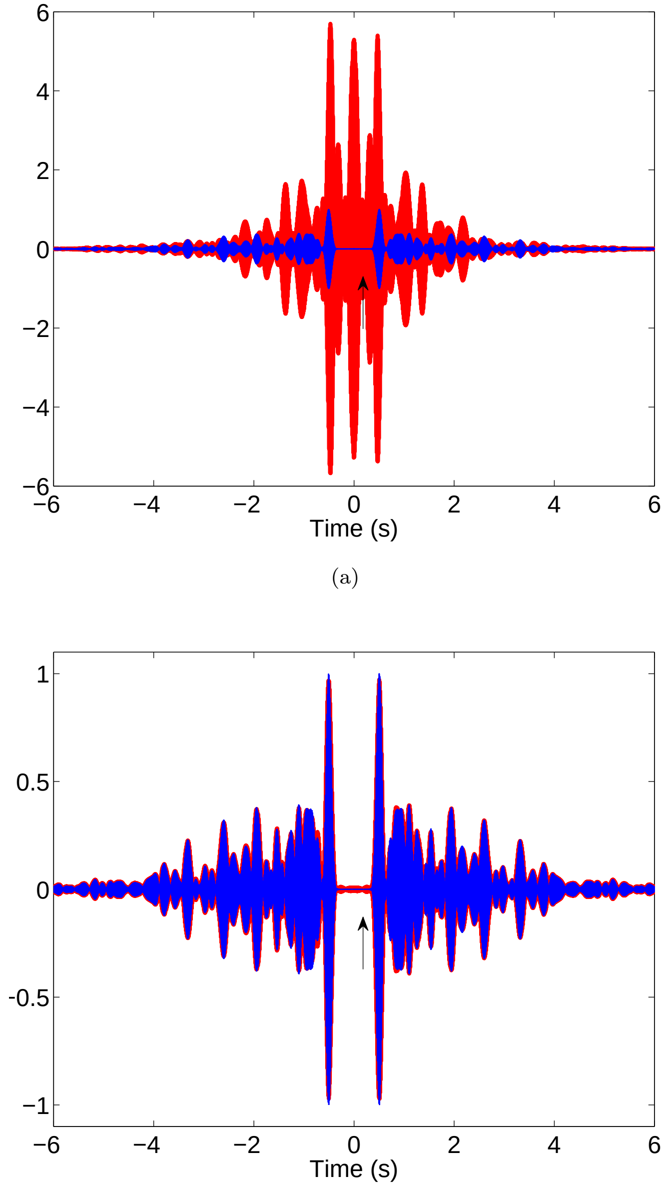


Figure 1. Two receivers, A and B, separated by a distance $d = 1.9$ km, are embedded in a two-dimensional acoustic scattering medium (unperturbed velocity $c_0 = 3.8$ km/s) characterized by n uniformly distributed isotropic point scatterers localized inside a circle of radius $r = 1.0$ km. A dense distribution of $N = 1000$ sources evenly spaced along a circle of radius $R = 4.0$ km surrounds the medium. For $n = 500$, the heterogeneous medium is considered strongly scattering. For $n = 10$, the scattering regime is weak.

for uncorrelated sources enclosing the studied system. This process, sometimes referred to as the *virtual source* method (Bakulin & Calvert 2006), is equivalent to having a source at one of the receiver locations. Further studies have extended the concept to a wide class of linear systems (Gouédard *et al.* 2008; Snieder *et al.* 2007; Wapenaar & Fokkema 2004; Wapenaar *et al.* 2006; Weaver 2008), and our work aims to accomplish the same objective.

We explore a general formulation of representation theorem for any system that satisfies a linear partial differential equation (or, mathematically, for any field in the appropriate Sobolev space). In particular, this formulation involves no assumption of spatial reciprocity or time-reversal invariance. We introduce a bilinear *interferometric operator* as a means of reconstructing the Green's function and study the influence of perturbations on the interferometric operator, and thereby derive a general representation theorem for perturbed media. The perturbed field can be retrieved by using a process characterized by the interferometric operation, which is generally more complex than cross-correlation. For common systems, this interferometric operation can be simplified using the symmetry properties of differential operators. We apply the theory to scattering problems and illustrate the approach with an example involving scattered acoustic waves, obtaining a result that concurs with that published by Vasconcelos *et al.* (2009) on the representation theorem for scattering in acoustic media. In geophysics, applications of perturbation reconstruction exist in the areas of, for example, crustal seismology, seismic imaging, well monitoring, and waveform inversion.

After exposing this general representation theorem for perturbed media, we give an innovative interpretation of Green's function reconstruction. To emphasize the connection between the general formulation and the particular case of scattering problems, we refer to unperturbed field as *direct field*, and field perturbation as *scattered field*. Perturbation retrieval can be understood in terms of interferences between unperturbed fields and field perturbations. One might think that field perturbations can be reconstructed with contributions from just field perturbations alone; the retrieval of field perturbations, however, requires the interferences with unperturbed fields. For acoustic media, this means that the *scattering response* between two receivers cannot be retrieved by cross-correlating only late coda waves. Here, the scattering response is defined as the superposition of the causal and acausal scattering Green's functions between the two points. In the numerical experiments conducted here (see Figure 1), two receivers are embed-



(a)

(b)

Figure 2. The blue curves show the actual *scattering response* (superposition of the causal and acausal scattering Green's functions) between two points embedded in a strongly heterogenous medium. The red curves represent the wave reconstructed by cross-correlating the waves recorded by two receivers at the same locations. Note the black arrow, which corresponds to the time of the first expected physical arrival. In panel (a), only scattered waves are cross-correlated. The reconstruction fails no matter how dense is the distribution of sources enclosing the medium. This failure of interferometry is not caused by restrictions of source distribution, aperture, or equipartitioning, but is a consequence of the missing contribution of recorded direct waves. In panel (b), both direct and scattered waves are cross-correlated, yielding a result confirming that the scattering response can be retrieved by interferometry.

ded in a scattering medium and surrounded by sources that are activated seperately, and consequently, generate uncorrelated wavefields. The numerical scheme is based on computation of the analytical solution to the two-dimensional heterogeneous acoustic wave equation for a distribution of isotropic point scatterers (Groenenboom & Snieder 1995). In Figure 2, we compare the actual scattering response for a source at the receiver location with the signal reconstructed by cross-correlating and summing the scattered waves recorded at the receiver positions. For a strongly scattering medium (average wavelength larger than several scattering mean free paths (Tourin *et al.* 2000)), Figure 2(a) shows that the reconstruction completely fails to retrieve the scattering response from cross-correlation of only the scattered waves recorded at the receiver locations. The reconstructed wave with only scattered waves is totally innaccurate: the early arrivals are non-physical because they do not respect causality, arriving before the minimum travel-time between the two receivers, while the late arrivals show no resemblance to the actual scattering response. Accurate retrieval of the scattered waves instead requires contributions from both direct and scattered waves, as shown in Figure 2(b).

In this paper, we provide an interpretation of this result; one can find a similar approach by Halliday & Curtis (2009) and Snieder & Fleury (2010), the latter of which describes the case of multiple scattering by discrete scatterers. In Snieder & Fleury (2010), we identify different scattering paths, show their contributions to the retrieval of either physical or nonphysical arrivals, and analyze how cancelations occur to allow the scattering Green's function to emerge. Our interpretation, along with that given by Halliday & Curtis (2009), leads to the same important conclusion: the cross-correlation of purely scattered waves does not allow extraction of the correct scattered waves.

The paper is organized as follows. In section 2, we describe the general systems under consideration and introduce the concept of perturbation. In section 3, we define the interferometric operator and its relation to representation theorems, emphasizing the influence of perturbations on this operator. Section 4 presents the general representation theorems for perturbations that follow this approach. In section 5, we apply this theory to interpret the reconstruction of Green's function perturbations; section 6 offers discussions and conclusions.

2 GREEN'S FUNCTION PERTURBATIONS FOR GENERAL SYSTEMS

Consider a general system governed by a linear partial differential equation in the frequency domain. In order to avoid the complexity of formalism that could obscure the main purpose of this paper, we leave the vector case for Appendix A. Let the complex scalar field $u_0(\mathbf{r}, \omega)$ be defined in a volume D_{tot}. One can adapt the result of this work to the time domain using the Fourier convention $u_0(\mathbf{r}, t) = \int u_0(\mathbf{r}, \omega) exp(-j\omega t) d\omega$. Henceforth, we suppress the frequency dependence of variables and operators. The unperturbed field $u_0(\mathbf{r})$ is a solution of the unperturbed equation

$$H_0(\mathbf{r}) \cdot u_0(\mathbf{r}) = s(\mathbf{r}), \tag{1}$$

where H_0 is the linear differential operator and s is the source term, associated with the unperturbed system. The dot denotes a contraction when vectors and tensors are considered. For acoustic waves, one may define H_0 as the propagator for non-uniform density media: $H_0 = \nabla \cdot (\rho_0^{-1} \nabla) + \rho_0^{-1} \omega^2 / c_0^2$, where ρ and c denote density and velocity, respectively.

Assuming a perturbation of the system, the perturbed field $u_1(\mathbf{r})$ follows from

$$H_1(\mathbf{r}) \cdot u_1(\mathbf{r}) \quad = \quad s(\mathbf{r}) \tag{2}$$

$$H_0(\mathbf{r}) \cdot u_1(\mathbf{r}) \quad = \quad V(\mathbf{r}) \cdot u_1(\mathbf{r}) + s(\mathbf{r}), \tag{3}$$

where V is the perturbation operator, and $H_1 = H_0 - V$ is the linear differential operator associated with the perturbed system. For example, for acoustic waves, with a change in velocity for the medium, the perturbation operator is $V = \rho_0^{-1} \omega^2 / c_0^2 (1 - c_0^2 / c_1^2)$. Alternatively, a change in experimental conditions might imply a variation in density; then, a way to account for this perturbation is to consider $V = (\rho_0^{-1} - \rho_1^{-1}) \omega^2 / c_0^2 + \nabla \cdot ((\rho_0^{-1} - \rho_1^{-1}) \nabla)$. One could also neglect attenuation in the medium in the first approximation and correct for it by introducing the perturbation $V = j\omega^2 \Im(\kappa_0^{-1} - \kappa_1^{-1})$, where $\Im$ denotes the imaginary part and $\kappa = \rho c^2$ is the bulk modulus. We are free to arbitrarily choose or even interchange the reference 0 and perturbed 1 states for any perturbation problem. Indeed, the perturbation need not necessarily introduce more complexity; its definition depends on the characteristics of the perturbation problem that one tries to solve.

For a problem to be well-defined, one needs to specify boundary conditions. Assume that the boundary conditions are unperturbed, and consider a regular problem with homogeneous boundary conditions:

$$B(\mathbf{r}) \cdot u_{0,1}(\mathbf{r}) = 0 \text{ on the boundary,} \tag{4}$$

where B denotes the linear boundary condition operator for the total volume D_{tot}. One can, for example, apply the Sommerfeld radiation condition for acoustic waves, but the boundary conditions need not be limited to being homogeneous. In Appendix B, we extend our reasoning to any unperturbed boundary conditions.

The Green's functions $G_0(\mathbf{r}, \mathbf{r}_S)$ and $G_1(\mathbf{r}, \mathbf{r}_S)$ for both unperturbed and perturbed systems are defined as solutions for an impulsive source at location $\mathbf{r}_S$,

$$s(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_S). \tag{5}$$

From the above equations, one obtains the familiar relation between unperturbed and perturbed Green's functions, known as the Lippmann-Schwinger equation (Rodberg & Thaler 1967):

$$G_1(\mathbf{r}, \mathbf{r}_S) = G_0(\mathbf{r}, \mathbf{r}_S) + \int_D G_0(\mathbf{r}, \mathbf{r}_1) \cdot V(\mathbf{r}_1) \cdot G_1(\mathbf{r}_1, \mathbf{r}_S) d^3 r_1, \tag{6}$$

where D is a subvolume of the total domain D_{tot}. We introduce the notation,

$$(f|g) \equiv \int_D f(\mathbf{r}) \cdot g(\mathbf{r}) d^3 r, \tag{7}$$

so that the Lippmann-Schwinger equation can be written as

$$G_1(\mathbf{r}, \mathbf{r}_S) = G_0(\mathbf{r}, \mathbf{r}_S) + \left(G_0(\mathbf{r}, \mathbf{r}_1) \left| V(\mathbf{r}_1) \right| G_1(\mathbf{r}_1, \mathbf{r}_S)\right). \tag{8}$$

Finally, we define the Green's function perturbation or *scattering Green's function* that characterizes the field perturbation $u_S(\mathbf{r}) = u_1(\mathbf{r}) - u_0(\mathbf{r})$ as

$$G_S(\mathbf{r}, \mathbf{r}_S) = G_1(\mathbf{r}, \mathbf{r}_S) - G_0(\mathbf{r}, \mathbf{r}_S), \tag{9}$$

or

$$G_S(\mathbf{r}, \mathbf{r}_S) = \left(G_0(\mathbf{r}, \mathbf{r}_1) \left| V(\mathbf{r}_1) \right| G_1(\mathbf{r}_1, \mathbf{r}_S)\right). \tag{10}$$

To clarify the terminology used throughout this paper, the unperturbed field, perturbed field, and field perturbation are denoted by u_0, u_1, and u_S, respectively.

3 DEFINITION OF THE INTERFEROMETRIC OPERATOR

To establish a representation theorem for perturbations, we first derive a general expression for Green's function retrieval by using a representation theorem of the correlation type (Wapenaar & Fokkema 2006). Consider two states of the field u, labeled A and B, governed by the partial differential equation $\mathcal{L}_{A,B}$,

$$\mathcal{L}_{A,B}: \quad H(\mathbf{r}) \cdot u_{A,B}(\mathbf{r}) = s_{A,B}(\mathbf{r}), \tag{11}$$

where the subscript $_{A,B}$ refers to either state A or B. Following Fokkema & Van den Berg (1993) and Fokkema *et al.* (1996), we evaluate $\left(u_A|\overline{\mathcal{L}}_B\right) - (\overline{u}_B|\mathcal{L}_A)$, where $\overline{f}$ denotes the complex conjugate of f; consequently,

$$\left(u_A \left|\overline{H}\right| \overline{u}_B\right) - (\overline{u}_B|H|u_A) = (u_A|\overline{s}_B) - (\overline{u}_B|s_A). \tag{12}$$

For impulsive sources, $s_{A,B}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_{A,B})$, and the fields $u_{A,B}(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_{A,B})$, the Green's functions in states A and B, so (12) becomes the general representation theorem of correlation-type for interferometry,

$$\begin{aligned} G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) &= \left(G(\mathbf{r}, \mathbf{r}_A) \left|\overline{H}(\mathbf{r})\right| \overline{G}(\mathbf{r}, \mathbf{r}_B)\right) \\ &- \left(\overline{G}(\mathbf{r}, \mathbf{r}_B) \left|H(\mathbf{r})\right| G(\mathbf{r}, \mathbf{r}_A)\right). \end{aligned} \tag{13}$$

This result is a general extension of the representation theorem in Snieder *et al.* (2007). In order to interpret and characterize the Green's function reconstruction more conveniently, we define the operator I_H,

$$I_H\{f, g\} \equiv \left(f \left|\overline{H}\right| g\right) - (g \left|H\right| f), \tag{14}$$

so that the general representation theorem can be written as

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = I_H\{G(\mathbf{r}, \mathbf{r}_A), \overline{G}(\mathbf{r}, \mathbf{r}_B)\}. \tag{15}$$

The operation $I_H\{\cdot, \cdot\}$ describes how Green's functions in a subvolume D "interfere" to reconstruct the Green's function between the two points A and B. We consequently refer to I_H as the *interferometric operator*, associated with H, that acts on functions f and g, and call the result of operation (14) an *interference* between f and g. For acoustic waves, the interferometric operation is the following volume integration:

$$I_H\{f, g\} = \int_D [f(\mathbf{r}) \nabla \cdot (\rho^{-1} \nabla g)(\mathbf{r}) - g(\mathbf{r}) \nabla \cdot (\rho^{-1} \nabla f)(\mathbf{r})] d^3 r. \tag{16}$$

Using Green's theorem, this volume integral becomes an integral over the bounding surface δD enclosing volume D:

$$I_H\{f, g\} = \oint_{\delta D} \rho^{-1}(\mathbf{r}) [f(\mathbf{r}) \nabla g(\mathbf{r}) - g(\mathbf{r}) \nabla f(\mathbf{r})] \cdot \hat{\mathbf{n}} d^2 r, \tag{17}$$

where $\hat{\mathbf{n}}$ is the outward unit normal vector at $\mathbf{r}$. Then, equation (15) retrieves the familiar representation theorem for acoustic waves (Wapenaar & Fokkema 2006):

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\delta D} \rho^{-1}(\mathbf{r})[G(\mathbf{r}, \mathbf{r}_A)(\mathbf{r})\nabla\overline{G}(\mathbf{r}, \mathbf{r}_B) - \overline{G}(\mathbf{r}, \mathbf{r}_B)\nabla G(\mathbf{r}, \mathbf{r}_A)] \cdot \hat{\mathbf{n}} d^2 r. \tag{18}$$

Returning to the general case, just as the unperturbed linear partial differential operator H_0 becomes $H_1 = H_0 - V$ after perturbating the system, the interferometric operators for unperturbed and perturbed systems, I_0 and I_1, relate in the following way:

$$\begin{aligned} I_0 &= I_{H_0} \\ I_1 &= I_0 - I_V. \end{aligned} \tag{19}$$

Note that, in general, I_0 and I_1 differ; that is, the interferometric operator is perturbed for a perturbed system. The exception ($I_1 = I_0$) occurs when $I_V = 0$. Consider, for example, the acoustic case previously described. The unperturbed Green's function is retrieved using expression (18), and, for a perturbation in velocity only,

$$I_V\{f, g\} = \int_D \frac{\omega^2}{\rho_0 {c_0}^2} \left[\left(1 - \left(\frac{c_0}{c_1}\right)^2\right) - \left(1 - \left(\frac{c_0}{c_1}\right)^2\right) \right] f(\mathbf{r}) g(\mathbf{r}) d^3 r = 0, \tag{20}$$

so $I_1 = I_0$. If, instead, density rather than velocity is perturbed,

$$I_V\{f, g\} = \oint_{\delta D} (\rho_0^{-1}(\mathbf{r}) - \rho_1^{-1}(\mathbf{r}))[f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})] \cdot \hat{\mathbf{n}} d^2 r \neq 0. \tag{21}$$

Therefore, the interferometric operator changes ($I_1 \neq I_0$) with such a perturbation. Similarly, with a perturbation in attenuation,

$$I_V\{f, g\} = j\omega^2 \int_D \Im(\kappa_0^{-1} - \kappa_1^{-1}) g(\mathbf{r}) f(\mathbf{r}) d^3 r \neq 0. \tag{22}$$

These examples illustrate that, in general, the same interferometric operation cannot be used to reconstruct both perturbed and unperturbed Green's functions; we need to estimate the perturbation of the interferometric operator, I_V, itself in order to apply interferometry for perturbed media. As seen in equations (21) and (22), the interferometric operator in general requires knowledge of medium properties for the perturbed system, a limiting factor because usually we know only the unperturbed medium properties. Equation (20), however, is a specific example of an interferometric operator that does remain unperturbed ($I_0 = I_1$) for nonzero perturbation. For benign cases such as this one, we need only know or estimate unperturbed medium properties, and measure or model both perturbed and unperturbed fields, in order to reconstruct the Green's functions.

Let us investigate such systems for which the interferometric operator is unperturbed ($I_V = 0$). Starting by reformulating the general representation theorem for both perturbed and unperturbed media, we retrieve the Green's functions using

$$G_{0,1}(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_{0,1}(\mathbf{r}_A, \mathbf{r}_B) = I_{0,1}\{G_{0,1}(\mathbf{r}, \mathbf{r}_A), \overline{G}_{0,1}(\mathbf{r}, \mathbf{r}_B)\}. \tag{23}$$

This expression clearly depends on the properties of the interferometric operator, and, according to definition (14), the reconstruction involves integration over the volume D. Because the integrand is a function of differential operators H_0 or H_1, and of the Green's functions between any point in D and points A or B, we need to know H_0, V, and the Green's functions for all points in the volume D in order to apply the interferometric operator and retrieve the Green's functions between A and B. In particular, estimation of the Green's functions for all points in D requires having sources throughout the entire volume D. To apply interferometry in practice, this requirement for sources or receivers over the entire volume is yet more limiting than the need to estimate perturbations of the medium properties; it would severely restrict the possibility of retrieving even unperturbed Green's functions.

In practice, we are interested in systems for which we can reconstruct Green's functions with a limited number of sources and receivers. Just as for acoustic waves in equation (18), we therefore aim for problems that enable us to transform the integration over volume D in expression (14) into integration over its boundary δD. This transformation allows significant reduction in the number of sources. In Appendix C, we show that this transformation can be done if and only if operators are *self-adjoint*. We also demonstrate that the self-adjoint symmetry of the operators implies spatial reciprocity under specific boundary conditions. In addition, the transformation of volume into surface integrals also constrains to just the surface δD the medium properties that must be known for the reconstruction. For perturbation problems that we are considering, we can always find a boundary of integration δD (for example, δD_{tot}) along which the system is unperturbed (there are no changes of the medium properties along δD). Then, under the assumption that H_0 and V are self-adjoint, the interferometric operation associated with this particular volume D can be reduced to an integration over δD, and the interferometric operator is then unperturbed under the assumption that the properties of the medium are unchanged along this boundary. Consequently, we can reconstruct the perturbed Green's function independently of the perturbations in the rest of the volume. For example, for a perturbation of densities in an acoustic medium, expression (21) illustrates that the interferometric operator is unperturbed ($I_V = 0$) when the density is unchanged on the boundary δD.

To summarize, interferometry can be interpreted as the application of an interferometric operator. This technique is practical for systems characterized by self-adjoint operators and for perturbation problems in which the interferometric operator is unperturbed.

4 REPRESENTATION FOR GREEN'S FUNCTION PERTURBATIONS

In the previous section, we established a general representation theorem for perturbed systems. Here, we derive a representation for field perturbations. This general representation differs from the traditional representation theorem for the special case of scattered acoustic waves (Vasconcelos *et al.* 2009) because, in general, we must take into account the perturbation of the interferometric operator. The perturbation of Green's function, defined in section 2, can be retrieved by interferometry by taking the difference of the two equations (23) for the perturbed and unperturbed states to give

$$G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) = I_1\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} - I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \tag{24}$$

Using relation (19) between unperturbed and perturbed interferometric operators, we have

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) &= I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} - I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &- I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}. \end{aligned} \tag{25}$$

Equation (25) is a general representation theorem for perturbation. Additionally, the interferometric operator is bilinear, i.e., $I_H\{\alpha f, g\} = I_H\{f, \alpha g\} = \alpha I_H\{f, g\}$, $I_H\{f, g+h\} = I_H\{f, g\} + I_H\{f, h\}$, and $I_H\{f+g, h\} = I_H\{f, h\} + I_H\{g, h\}$. We exploit the bilinearity of I_0 and expand $I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$ in terms of unperturbed fields and field perturbations:

$$\begin{aligned} I_0\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\} &= I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} \\ &+ I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}. \end{aligned} \tag{26}$$

This decomposition allows for the identification of different types of interference between unperturbed Green's functions and Green's function perturbations. Then, inserting equation (26) into representation theorem (25), gives

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) &= I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} \\ &+ I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} - I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}. \end{aligned} \tag{27}$$

Representation theorem (27) illustrates that the retrieval of Green's function perturbations requires a combination of interferences between both unperturbed Green's functions and Green's function perturbations. In section 5, we analyze the individual contributions of the different terms on the right-hand side of equation (27) to the reconstruction. Notice in particular the term $I_V\{G_1(\mathbf{r}, \mathbf{r}_A), \overline{G}_1(\mathbf{r}, \mathbf{r}_B)\}$, which represents the interference between perturbed Green's functions associated with the operator V, and accounts for the perturbation of the interferometric operator. Where possible, we prefer to consider situations for which $I_V = 0$ because in such cases,

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B) &= I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + I_0\{G_0(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\} \\ &+ I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_S(\mathbf{r}, \mathbf{r}_B)\}. \end{aligned} \tag{28}$$

Representation theorem (28) is a function of only the unperturbed interferometric operator I_0, and, consequently, depends only on the properties of the unperturbed medium. For these special cases, such as acoustic waves with velocity perturbation, the preturbation retrieval does not require an estimation of the perturbation V.

Now, let us return to the general case, $I_V \neq 0$, and establish another form of representation theorem for perturbations, one that characterizes only the causal Green's function perturbation, $G_S(\mathbf{r}_B, \mathbf{r}_A)$, rather than the superposition of the causal and acausal functions, $G_S(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A, \mathbf{r}_B)$. This representation will help in analyzing the individual contribution of the interference between direct and scattered fields to the partial retrieval of the scattered field $G_S(\mathbf{r}_B, \mathbf{r}_A)$. Rearranging relation (23) for unperturbed systems and inserting it into equation (10) yields

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= \left([I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + \overline{G}_0(\mathbf{r}_1, \mathbf{r}_B)] \, |V(\mathbf{r}_1)| \, G_1(\mathbf{r}_1, \mathbf{r}_A) \right) \\ &= I_0\{ (G_0(\mathbf{r}, \mathbf{r}_1) \, |V(\mathbf{r}_1)| \, G_1(\mathbf{r}_1, \mathbf{r}_A)) \, , \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + \left(\overline{G}_0(\mathbf{r}_1, \mathbf{r}_B) \, |V(\mathbf{r}_1)| \, G_1(\mathbf{r}_1, \mathbf{r}_A) \right). \end{aligned} \tag{29}$$

Using once again expression (10), which defines the Green's function perturbation, we identify the first term on the right-hand side of (29) with $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\}$ to obtain

$$G_S(\mathbf{r}_B,\mathbf{r}_A) = I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\} + \left(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\right). \tag{30}$$

This representation theorem for perturbations generalizes to any physical system the representation theorem for the special case of acoustic waves (Vasconcelos *et al.* 2009),

$$\begin{aligned} G_S(\mathbf{r}_B,\mathbf{r}_A) &= \oint_{\delta D} \rho_0^{-1}(\mathbf{r})[G_S(\mathbf{r},\mathbf{r}_A)\nabla\overline{G}_0(\mathbf{r},\mathbf{r}_B) - \overline{G}_0(\mathbf{r},\mathbf{r}_B)\nabla G_S(\mathbf{r},\mathbf{r}_A)]\cdot\hat{\mathbf{n}}d^2r \\ &+ \int_D \overline{G}_0(\mathbf{r},\mathbf{r}_B)V(\mathbf{r})G_1(\mathbf{r},\mathbf{r}_A)d^3r. \end{aligned} \tag{31}$$

Representation theorems (25) and (30) offer the possibility of extracting field perturbations (e.g., scattered waves) between points A and B as if one of these points acts as a source. They allow calculation of perturbation propagation between these two points without the need for a physical source at either of the two locations. These representation theorems have potential for estimating not only perturbations in fields but perturbations in medium properties by treating expression (30) as an integral equation for the perturbation V given the field perturbation G_S. They can therefore be used for detecting, locating, monitoring, and modeling medium perturbations. In geoscience, this theory has application to a diversity of techniques including passive imaging using seismic noise, seismic migration, modeling for inversion of electromagnetic data, and remote monitoring of hydrocarbon reservoirs, aquifers, and CO_2 injection for carbon sequestration.

5 ANALYSIS OF THE DIFFERENT CONTRIBUTIONS TO THE RETRIEVAL OF PERTURBATIONS

Here, we analyze the different terms that contribute to representation theorem (27) for perturbations. In particular, we interpret the contribution of the interference between field perturbations, corresponding to the term $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$, and explain why perturbations cannot be reconstructed by using solely the interference between perturbations; that is, the reconstruction of perturbations requires knowledge of the unperturbed fields for the system. We show that the contribution of the interference between unperturbed fields and field perturbations, corresponding to the terms $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\}$ and $I_0\{G_0(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$, is responsible for retrieving only field perturbations that are contaminated by spurious events. The interference between just the field perturbations is necessary to cancel these contaminants. To a certain extent, the cancelation mechanism involved in the reconstruction process can be connected to the general optical theorem as discussed below.

5.1 Partial retrieval of field perturbations

First, consider the contributions of the interferences between unperturbed fields and field perturbations. Rearranging the terms in representation theorem (30), we have the two following expressions, equation (33) being the negative conjugate of equation (32):

$$I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\} = G_S(\mathbf{r}_B,\mathbf{r}_A) - \left(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\right), \tag{32}$$

$$I_0\{G_0(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} = -\overline{G}_S(\mathbf{r}_A,\mathbf{r}_B) + \left(G_0(\mathbf{r},\mathbf{r}_A)\,\big|\overline{V}(\mathbf{r})\big|\,\overline{G}_1(\mathbf{r},\mathbf{r}_B)\right). \tag{33}$$

Equations (32) and (33) show that the terms $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\}$ and $I_0\{G_0(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$ contribute to the causal and acausal Green's function perturbation between A and B, respectively. Note, however, the two additional volume integrals that depend on the perturbation operator:
$\left(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\right)$ and $\left(G_0(\mathbf{r},\mathbf{r}_A)\,\big|\overline{V}(\mathbf{r})\big|\,\overline{G}_1(\mathbf{r},\mathbf{r}_B)\right)$. Their presence thus contaminates the estimate of the Green's function perturbation with spurious contributions (called *spurious arrivals* by Snieder *et al.* (2008)). In general, we cannot neglect them because they do not vanish regardless of the subspace D under consideration. Depending on the perturbation V, however, these spurious contributions can be relatively small. The summation of equations (32) and (33) thus gives a retrieval of the field perturbation, $G_S(\mathbf{r}_B,\mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A,\mathbf{r}_B)$, contaminated with spurious arrivals.

To get insight into the physical meaning of this partial reconstuction, let us particularize the general description of equations (32) and (33) to the case of acoustic waves in which direct waves interfere with scattered waves. Figure 3 illustrates the reconstruction obtained by cross-correlating just direct and scattered waves for both weakly and strongly scattering media (Figures 3(a) and 3(b), respectively). Interestingly, for a weakly scattering medium (average

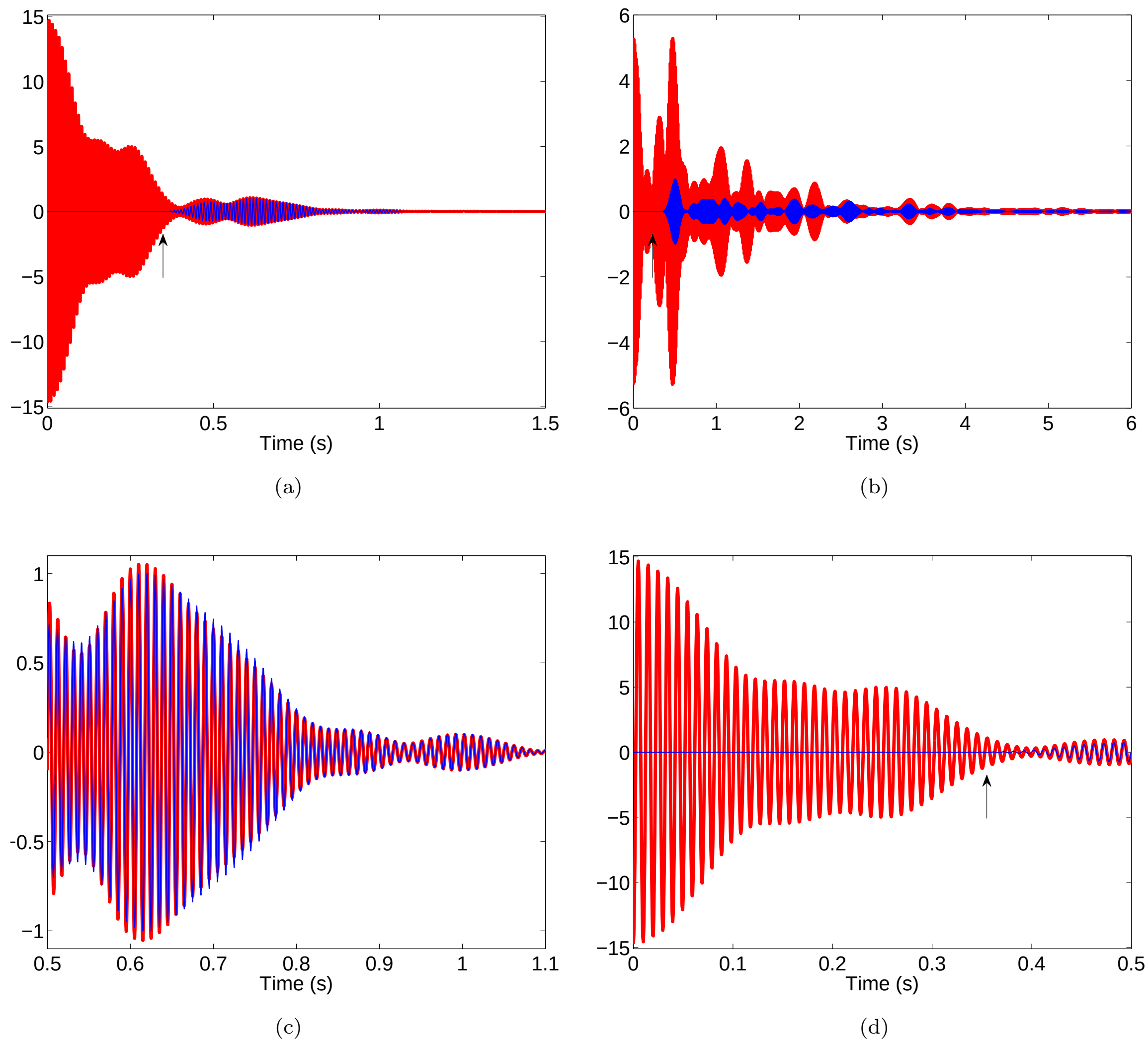


Figure 3. The causal part of the actual *scattering response* (blue curves) between two points embedded in heterogeneous media is compared to the reconstructed wave (red curves) obtained by cross-correlating direct and scattered waves recorded by two receivers at the same locations. Panels (a) and (b) show the signals for a weakly and strongly scattering medium, respectively. Panel (c) and (d) provide zooms on the late and early parts of experiment in weakly scattering regime, respectively. In both scattering regimes, the reconstruction is inaccurate. The weakly scattering case, however, suggests a partial retrieval of the scattering response: the reconstructed and reference signals are similar in their late parts (Panel (c)) while the early part of the reconstructed signal (i.e., the portion before the time of the direct arrival, denoted by the arrow) is purely erroneous (Panel (d)) and contains only the spurious arrivals.

wavelength less or about the scattering mean free path), Figure 3(c) shows a reconstructed signal that fully retrieves the late portion of the scattering response. The early part of the signal, however, contains strong nonphysical arrivals, prior to the true first arrival (arrow), as seen in Figure 3(d). These observations suggest that while the signal reconstructed by cross-correlating direct and scattered waves does contain the scattering response, it is contaminated by spurious arrivals. Figure 3(a) shows that, for a strongly scattering medium, the reconstructed signal is contaminated so severely that no similarities can be found between the reconstructed and reference signals; the contribution of the spurious arrivals dominates the reconstruction. In summary, because the physical nature of the spurious arrival is the same for both weakly and strongly scattering media, cross-correlating direct and scattered waves retrieves the scattered waves but generates unexpected arrivals that can be more intense than the useful signal. These spurious arrivals must cancel in order for the retrieval of scattered waves to be completed.

5.2 Cancelation of the spurious arrivals

The interference between direct and scattered waves, i.e., the first two terms in (27), partially retrieves the scattered waves. We are interested in studying the mechanism for canceling the spurious arrivals described in the previous subsection. According to representation theorem (27), completion of the reconstruction requires the additional contributions from the interferences $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$, and $I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}$. In the introduction, we showed numerically that the interference between scattered waves alone does not correctly retrieve scattered waves. Taken individually, the interference between unperturbed fields and field perturbations, $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\}$ and $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_0(\mathbf{r},\mathbf{r}_B)\}$, the interference between just the field perturbations $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$, or the interference $I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}$ does not reconstruct field perturbations. The summation of all their contributions, however, is expected to accurately retrieve the perturbations and, consequently, cancel the spurious arrivals.

We develop the following relation for the interference between field perturbations by rewriting $I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}$:

$$\begin{aligned}
I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} &= I_0\{(G_0(\mathbf{r},\mathbf{r}_1)\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A))\,,(\overline{G}_0(\mathbf{r},\mathbf{r}_2)\,|\overline{V}(\mathbf{r}_2)|\,\overline{G}_1(\mathbf{r}_2,\mathbf{r}_B))\}\\
&= \big(\big(I_0\{G_0(\mathbf{r},\mathbf{r}_1),\overline{G}_0(\mathbf{r},\mathbf{r}_2)\}\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A)\big)\,|\overline{V}(\mathbf{r}_2)|\,\overline{G}_1(\mathbf{r}_2,\mathbf{r}_B)\big)\\
&= \big(\big([G_0(\mathbf{r}_2,\mathbf{r}_1)-\overline{G}_0(\mathbf{r}_1,\mathbf{r}_2)]\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A)\big)\,|\overline{V}(\mathbf{r}_2)|\,\overline{G}_1(\mathbf{r}_2,\mathbf{r}_B)\big)\\
&= \big((G_0(\mathbf{r}_2,\mathbf{r}_1)\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A))\,|\overline{V}(\mathbf{r}_2)|\,\overline{G}_1(\mathbf{r}_2,\mathbf{r}_B)\big)\\
&- \big((\overline{G}_0(\mathbf{r}_1,\mathbf{r}_2)\,|\overline{V}(\mathbf{r}_2)|\,\overline{G}_1(\mathbf{r}_2,\mathbf{r}_B))\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A)\big)\,.
\end{aligned} \tag{34}$$

Here, we used expression (10) for field perturbations in the first identity, the bilinearity of I_0 in the second identity, and representation theorem (23) in the third identity; so that finally,

$$I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} = \big(G_S(\mathbf{r}_1,\mathbf{r}_A)\,|\overline{V}(\mathbf{r}_1)|\,\overline{G}_1(\mathbf{r}_1,\mathbf{r}_B)\big) - \big(\overline{G}_S(\mathbf{r}_1,\mathbf{r}_B)\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1,\mathbf{r}_A)\big)\,. \tag{35}$$

We next show that the interaction between Green's function perturbations indirectly retrieves the Green's function perturbation by contributing to the cancelation of the spurious arrivals. The right-hand side of equation (35) is the complement of the spurious contributions $-\big(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\big)$ and $\big(G_0(\mathbf{r},\mathbf{r}_A)\,|\overline{V}(\mathbf{r})|\,\overline{G}_1(\mathbf{r},\mathbf{r}_B)\big)$ in equations (32) and (33); that is, the summation of these integrals retrieves the term $-I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}$. For cases in which $I_V=0$, the interaction between perturbations entirely cancels the spurious arrivals,

$$I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} + \big(G_0(\mathbf{r},\mathbf{r}_A)\,|\overline{V}(\mathbf{r})|\,\overline{G}_1(\mathbf{r},\mathbf{r}_B)\big) - \big(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\big) = 0, \tag{36}$$

and the reconstruction is then completed by summing the contributions from equations (32), (33) and (35) (the sum reduces to representation theorem (28)). For the general case ($I_V \neq 0$),

$$\begin{aligned}
I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} + \big(G_0(\mathbf{r},\mathbf{r}_A)\,|\overline{V}(\mathbf{r})|\,\overline{G}_1(\mathbf{r},\mathbf{r}_B)\big) - \big(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r},\mathbf{r}_A)\big)&\\
= -I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\},&
\end{aligned} \tag{37}$$

and the summation of equations (32), (33) and (35) gives

$$(32)+(33)+(35) = G_S(\mathbf{r}_B,\mathbf{r}_A) - \overline{G}_S(\mathbf{r}_A,\mathbf{r}_B) + I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}. \tag{38}$$

The retrieval is incomplete and does not produce the Green's function perturbation because of the term $I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}$ that still contaminates the right-hand side of equation (38). Accurate reconstruction requires an additional estimate of this interaction between perturbed fields associated with V.

In any case, a direct consequence for scattering problems is that we cannot reconstruct the scattering Green's function by merely using the contribution of scattered waves alone. This explains the failure of interferometry based solely on the interference of scattered waves, as shown in Figure 2. The interference between Green's function perturbations nevertheless plays a fundamental role in the retrieval of the perturbation because they are needed to cancel spurious arrivals. Our numerical experiments illustrate this observation for scattered acoustic waves (Figure 4). For both weakly and strongly scattering media, combining the contributions of both interference between direct and scattered waves and interference between just scattered waves cancels the spurious arrivals and reconstructs the superposition of the causal and acausal scattering Green's functions. Note, additionally, that in order for this experiment to be successful, the distribution of sources must be sufficiently dense on a close surface surrounding the receivers (see numerical set-up description in Figure 1). Considerations of narrow aperture and limited number of sources are independent problems that limit the accuracy of reconstructions (Fan & Snieder 2009; Snieder 2004).

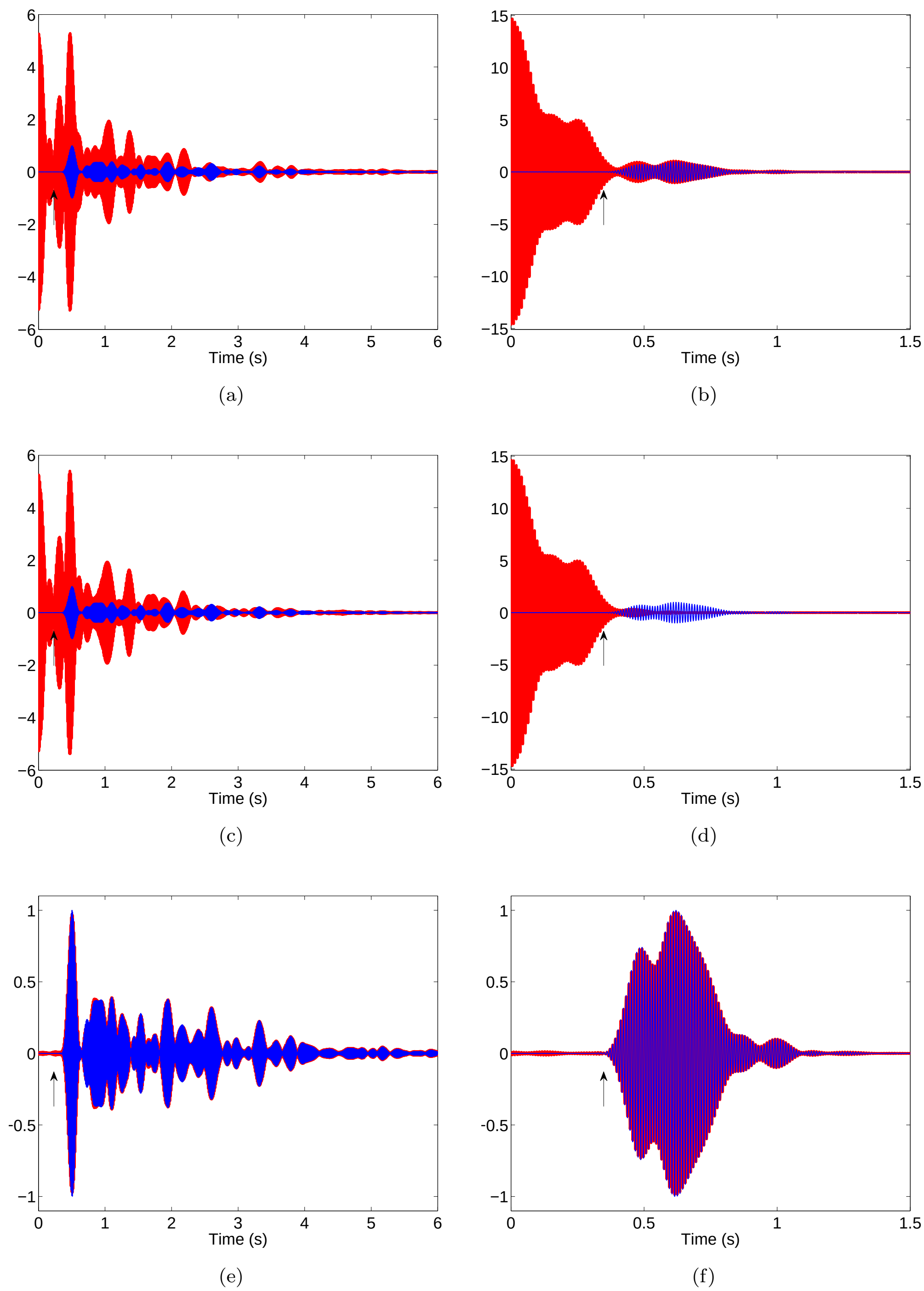


Figure 4. The blue curves show the causal part of the *scattering response* between two points embedded in heterogeneous acoustic media. The red curves correspond to the reconstructed signals for the different individual contributions discussed in section 5. For strongly scattering media (left column), the summation of the reconstructed signal by cross-correlating direct and scattered waves (a) with that obtained by cross-correlating scattered waves (c) leads to the retrieval of the scattering response and cancelation of the spurious arrivals (e). Likewise, (b), (d), and (f) show success of the reconstruction for weakly scattering media(right column).

5.3 Connection with the general optical theorem

Above, we emphasize the process that leads to the reconstruction of perturbations. Interestingly, for problems with unperturbed interferometric operators, the interference between field perturbations alone contributes entirely to the cancelation of the spurious arrivals that arise from the interferences between unperturbed fields and field perturbations in the reconstruction process, and rewriting equation (36) gives

$$\left(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\left|V(\mathbf{r})\right|G_1(\mathbf{r},\mathbf{r}_A)\right) - \left(G_0(\mathbf{r},\mathbf{r}_A)\left|\overline{V}(\mathbf{r})\right|\overline{G}_1(\mathbf{r},\mathbf{r}_B)\right) = I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\}. \tag{39}$$

In a sense, we can interpret this mechanism as an extension of the general optical theorem, as has been suggested for acoustic waves (Snieder *et al.* 2009, 2008). The general optical theorem (Marston 2001; Schiff 1968) concerns the scattering amplitude $f_k(\hat{\mathbf{n}},\hat{\mathbf{n}'})$ of scattered waves with wave number k, and unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}'}$ representing the directions of the outgoing and incoming waves, respectively. With a far-field approximation in expression (17), the interferometric operator for the constant-density wave equation ($\rho_0 = 1$) becomes

$$I_0\{f,g\} = 2jk\oint_{\delta D} f(\mathbf{r})g(\mathbf{r})d^2r \tag{40}$$

for a homogenous medium as the unperturbed state ($G_0(\mathbf{r},\mathbf{r}_S) = \dfrac{e^{-jk\|\mathbf{r}-\mathbf{r}_S\|}}{4\pi\|\mathbf{r}-\mathbf{r}_S\|}$). With the medium perturbed by a single scattering object positioned at $\mathbf{r}_x$, the scattering Green's function in the far field is given by

$$G_S(\mathbf{r},\mathbf{r}_S) = 4\pi G_0(\mathbf{r},\mathbf{r}_x)f_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_S)G_0(\mathbf{r}_x,\mathbf{r}_S). \tag{41}$$

If A and B are far from the scatterer and δD is a sphere centered at $\mathbf{r}_x$ with radius R, the interference between scattered Green's functions is

$$I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} = 2jk\oint_{\delta D} G_0(\mathbf{r}_x,\mathbf{r}_A)\overline{G}_0(\mathbf{r}_x,\mathbf{r}_B)f_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_B)(4\pi)^2G_0(\mathbf{r},\mathbf{r}_x)\overline{G}_0(\mathbf{r},\mathbf{r}_x)d^2r. \tag{42}$$

The integration over the sphere δD is equivalent to an integration over solid angle by $d^2r = R^2d\hat{n}$, and $(4\pi)^2G_0(\mathbf{r},\mathbf{r}_x)\overline{G}_0(\mathbf{r},\mathbf{r}_x) = \dfrac{1}{R^2}$ so that

$$I_0\{G_S(\mathbf{r},\mathbf{r}_A),\overline{G}_S(\mathbf{r},\mathbf{r}_B)\} = 2jkG_0(\mathbf{r}_x,\mathbf{r}_A)\overline{G}_0(\mathbf{r}_x,\mathbf{r}_B)\oint f_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_B)d\hat{n}. \tag{43}$$

In the far-field approximation for the scattering Green's function, one can modify previously established equations by using expression (41) instead of (10) for the field perturbation. Consequently, the spurious contributions introduced in equations (32) and (33) are

$$\left(\overline{G}_0(\mathbf{r},\mathbf{r}_B)\left|V(\mathbf{r})\right|G_1(\mathbf{r},\mathbf{r}_A)\right) = 4\pi\overline{G}_0(\mathbf{r}_x,\mathbf{r}_B)f_k(\hat{\mathbf{n}}_B,\hat{\mathbf{n}}_A)G_0(\mathbf{r}_x,\mathbf{r}_A), \tag{44}$$

$$\left(G_0(\mathbf{r},\mathbf{r}_A)\left|\overline{V}(\mathbf{r})\right|\overline{G}_1(\mathbf{r},\mathbf{r}_B)\right) = 4\pi G_0(\mathbf{r}_x,\mathbf{r}_A)\overline{f}_k(\hat{\mathbf{n}}_A,\hat{\mathbf{n}}_B)\overline{G}_0(\mathbf{r}_x,\mathbf{r}_B), \tag{45}$$

and we thus retrieve the general optical theorem from equation (39):

$$f_k(\hat{\mathbf{n}}_B,\hat{\mathbf{n}}_A) - \overline{f}_k(\hat{\mathbf{n}}_A,\hat{\mathbf{n}}_B) = \frac{2jk}{4\pi}\oint f_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_A)\overline{f}_k(\hat{\mathbf{n}},\hat{\mathbf{n}}_B)d\hat{n}. \tag{46}$$

This interpretation of the cancelations, however, is limited to problems with unperturbed interferometric operators. For general systems, the spurious arrivals do not cancel by summing the interferences associated with the unperturbed operator H_0. Unless the interferometric operator is unperturbed ($I_V = 0$), the interference associated with V on the right-hand side of equation (38) still contaminates the perturbations we desire to reconstruct by adding the contributions from equations (32), (33) and (35). In general, we have to evaluate the contribution of $I_V\{G_1(\mathbf{r},\mathbf{r}_A),\overline{G}_1(\mathbf{r},\mathbf{r}_B)\}$ in order to cancel the spurious arrivals and reconstruct the exact field perturbations. Thus as stated in section 3, because the perturbation operator is usually unknown, interferometry appears practical for perturbation problems only with an interferometric operator that is unperturbed.

In summary, we have shown that the scattering response cannot be retrieved by cross-correlating scattered waves alone. To reconstruct scattered waves, we need to consider the contribution from cross-correlation of direct and scattered waves. The key to the ability to cancel the spurious arrivals and succeed in the reconstruction for any kind of perturbation problem is that we consider systems for which the interferometric operator is unperturbed, $I_V = 0$.

6 DISCUSSION AND CONCLUSION

We have derived a representation theorem for general systems and in particular for perturbed media. This makes it possible to retrieve Green's functions and their perturbations for a large variety of linear differential systems that include acoustic, elastic, and electromagnetic waves. We show the extension to vector fields in Appendix A. We investigate the reconstruction of Green's functions, applying an interferometric operator to unperturbed fields and field perturbations. This mathematical description of interferometry simplifies the analysis of the reconstruction of perturbations: we interpret this process as summing contributions from different types of interference between perturbations and unperturbed Green's functions. In geophysics, this description can be applied to a range of problems. For example, one can extend conventional interferometry techniques for seismic waves to some possible applications in imaging and inverse problems: the representation theorem can be related to sensitivity kernels used in waveform inversion (Tarantola & Tarantola 1987), in imaging (Colton & Kress 1998), or in tomography (Woodward 1992); the theorem also allows the establishment of formal connections with seismic migration (Clearbout 1985) and with inverse scattering methods (Beylkin 1985; Borcea *et al.* 2002).

Our study of the retrieval of perturbations differs from previous work because we show explicitly that not only fields are perturbed but the operator itself changes when the medium is perturbed. For most general systems, we would need to modify the interferometric process used for the reconstruction after the application of a perturbation. We obtain this fundamental result after deriving the perturbation of the interferometric operator. Our analysis emphasizes the importance of those systems for which the interferometric operator is unperturbed because such systems appear to offer the prospect for practical application of interferometry. In these cases, reconstruction of the Green's function perturbations does not require knowledge or estimation of the perturbations of the medium properties. We also demonstrate that perturbations cannot be retrieved by measuring only field perturbations; knowledge of the unperturbed state of the studied system is essential as well. Perturbations are reconstructed by combining interferences between field perturbations and unperturbed fields. The contribution from interference of field perturbations alone cancels the erroneous arrivals generated by interference of unperturbed fields with field perturbations.

Simulations for scattering acoustic media show the importance of direct arrivals in the extraction of scattering responses and verify the failure to reconstruct scattering Green's function by cross-correlating just scattered waves. This result is intriguing and should be carefully considered when designing applications because our result appears to be in contradiction to many results in seismology. Campillo & Paul (2003), for example, have shown that cross-correlation of just late coda in earthquake data, allows for retrieval of direct surface waves. Also, Stehly *et al.* (2008) have used the coda of the cross-correlation of seismic noise for improving the reconstruction of Green's functions. Indeed, the main components of late coda waves are scattered waves. So, what might be the source of this apparent discrepancy with our results? We base our reasoning on interpretations of representation theorems for perturbed systems, and study the extraction of scattered waves without performing any time averaging as is done in the work published in these papers. Further work needs to be done to explore the hypothesis that it could be the averaging that allows reconstruction from scattered waves alone. Perhaps what is being reconstructed by the time averaging in those papers is just some component of the Green's function, or some average Green's function, not the Green's function itself. In geoscience, Campillo & Paul (2003), Halliday & Curtis (2008), Roux *et al.* (2005), and Shapiro *et al.* (2005) have shown that direct surface waves are beautifully extracted by interferometry; but examples of reconstruction of scattered surface and body waves are lacking. Again, the general formulation of the representation theorem for perturbed media states that we can in principle retrieve any and all perturbations for a given system.

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APPENDIX A: EXTENSION TO VECTOR SPACES AND $N \times N$ DIFFERENTIAL OPERATORS

Here, we extend our reasoning to vector fields by using the tensor notation previously introduced. Consider the unperturbed field $\boldsymbol{u}_0(\mathbf{r})$, defined in the vector space D_{tot} of dimension n, which is a solution of equation

$$\boldsymbol{H}_0(\mathbf{r}) \cdot \boldsymbol{u}_0(\mathbf{r}) = \boldsymbol{s}(\mathbf{r}), \tag{A1}$$

where $\boldsymbol{H}_0$ and $\boldsymbol{s}$ are the $n \times n$ linear differential operator and the source term, respectively. For elastic waves, the propagator is

$$\boldsymbol{H}_0 = \rho\omega^2 \boldsymbol{\delta} + \boldsymbol{\nabla} \cdot \boldsymbol{c} \cdot \boldsymbol{\nabla}, \tag{A2}$$

where $\boldsymbol{c}$ is the elasticity tensor and $\boldsymbol{\delta}$ the Kronecker tensor; $\boldsymbol{u}_0$ is the displacement vector. For electromagnetic waves in isotropic media (ϵ, permittivity; σ, conductivity; μ, permeability), the operator is

$$\boldsymbol{H}_0 = \begin{bmatrix} \boldsymbol{\nabla}\times & -j\omega\mu\boldsymbol{\delta} \\ (j\omega\epsilon - \sigma)\boldsymbol{\delta} & \boldsymbol{\nabla}\times \end{bmatrix} \text{with } \boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{e}_0 \\ \boldsymbol{h}_0 \end{bmatrix}, \tag{A3}$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ denote the electric and magnetic fields, respectively. We give two examples of systems for which our reasoning applies; futher cases of study can be found in Wapenaar *et al.* (2006). The perturbed field $\boldsymbol{u}_1(\mathbf{r})$ satisfies

$$\boldsymbol{H}_0(\mathbf{r}) \cdot \boldsymbol{u}_1(\mathbf{r}) = \boldsymbol{V}(\mathbf{r}) \cdot \boldsymbol{u}_1(\mathbf{r}) + \boldsymbol{s}(\mathbf{r}), \tag{A4}$$

where $\boldsymbol{V}$ is the perturbation operator. Elastic waves can be perturbed in the presence of viscosity ($\boldsymbol{\eta}$ tensor), in which case we write $\boldsymbol{V}$ as

$$\boldsymbol{V} = -j\omega\boldsymbol{\nabla} \cdot \boldsymbol{\eta} \cdot \boldsymbol{\nabla}. \tag{A5}$$

A change in medium properties ($\delta\epsilon$, $\delta\sigma$, $\delta\mu$) influences electromagnetic waves by a perturbation

$$\boldsymbol{V} = \begin{bmatrix} 0 & j\omega\delta\mu\boldsymbol{\delta} \\ (\delta\sigma - j\omega\delta\epsilon)\boldsymbol{\delta} & 0 \end{bmatrix}. \tag{A6}$$

Assume a regular problem with unperturbed homogeneous boundary conditions. We relate the Green's tensors $\boldsymbol{G}_1(\mathbf{r}, \mathbf{r}_S)$ and $\boldsymbol{G}_0(\mathbf{r}, \mathbf{r}_S)$ by using the Lippmann-Schwinger equation:

$$\boldsymbol{G}_1(\mathbf{r}, \mathbf{r}_S) = \boldsymbol{G}_0(\mathbf{r}, \mathbf{r}_S) + \left(\boldsymbol{G}_0(\mathbf{r}, \mathbf{r}_1) \left| \boldsymbol{V}(\mathbf{r}_1) \right| \boldsymbol{G}_1(\mathbf{r}_1, \mathbf{r}_S)\right); \tag{A7}$$

let the perturbation of the Green's tensor $\boldsymbol{G}_S(\mathbf{r}, \mathbf{r}_S)$ be given by $\boldsymbol{G}_S(\mathbf{r}, \mathbf{r}_S) = \boldsymbol{G}_1(\mathbf{r}, \mathbf{r}_S) - \boldsymbol{G}_0(\mathbf{r}, \mathbf{r}_S)$.

The new bilinear interferometric operator $I_{\boldsymbol{H}}$ now acts on matrices,

$$I_{\boldsymbol{H}}\{\boldsymbol{F}, \boldsymbol{G}\} = \left(\boldsymbol{F}^T \left|\overline{\boldsymbol{H}}\right| \boldsymbol{G}\right) - \left(\boldsymbol{G}^T \left|\boldsymbol{H}\right| \boldsymbol{F}\right), \tag{A8}$$

where $\boldsymbol{F}^T$ denotes the transpose of the matrix $\boldsymbol{F}$. We introduce the unperturbed and perturbed interferometric operators as

$$\begin{aligned} I_0\{\boldsymbol{F}, \boldsymbol{G}\} &= I_{\boldsymbol{H}_0}\{\boldsymbol{F}, \boldsymbol{G}\} \\ I_1\{\boldsymbol{F}, \boldsymbol{G}\} &= I_{\boldsymbol{H}_0}\{\boldsymbol{F}, \boldsymbol{G}\} - I_{\boldsymbol{V}}\{\boldsymbol{F}, \boldsymbol{G}\}. \end{aligned} \tag{A9}$$

Consequently, the general representation theorem for vector systems becomes

$$\boldsymbol{G}_{0,1}(\mathbf{r}_B, \mathbf{r}_A) - \boldsymbol{G}_{0,1}^H(\mathbf{r}_A, \mathbf{r}_B) = I_{0,1}\{\boldsymbol{G}_{0,1}(\mathbf{r}, \mathbf{r}_A), \overline{\boldsymbol{G}}_{0,1}(\mathbf{r}, \mathbf{r}_B)\}, \tag{A10}$$

where $\boldsymbol{G}_{0,1}^H$ denotes the hermitian conjugate of $\boldsymbol{G}_{0,1}$. For elastic waves,

$$I_0\{\boldsymbol{F}, \boldsymbol{G}\} = \oint_{\delta D} \left(\boldsymbol{F}^T(\mathbf{r}) \cdot \boldsymbol{c}(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{G}(\mathbf{r}) - \boldsymbol{G}^T(\mathbf{r}) \cdot \boldsymbol{c}(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{F}(\mathbf{r})\right) \cdot \hat{\mathbf{n}} d^2 r \tag{A11}$$

and

$$I_{\boldsymbol{V}}\{\boldsymbol{F}, \boldsymbol{G}\} = j\omega \int_D \left(\boldsymbol{F}^T(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\eta}(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{G}(\mathbf{r}) + \boldsymbol{G}^T(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\eta}(\mathbf{r}) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{F}(\mathbf{r})\right) d^3 r. \tag{A12}$$

For electromagnetic waves, we derive

$$\begin{aligned} I_0\{\boldsymbol{F},\boldsymbol{G}) &= \oint_{\delta D} \boldsymbol{G}^T(\mathbf{r}) \cdot \begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix} \cdot \boldsymbol{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} d^2 r \\ &+ \int_D \boldsymbol{F}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & j\omega\mu\boldsymbol{\delta} \\ -(\sigma + j\omega\epsilon)\boldsymbol{\delta} & 0 \end{bmatrix} \cdot \boldsymbol{G}(\mathbf{r}) d^3 r \\ &+ \int_D \boldsymbol{G}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & j\omega\mu\boldsymbol{\delta} \\ (\sigma - j\omega\epsilon)\boldsymbol{\delta} & 0 \end{bmatrix} \cdot \boldsymbol{F}(\mathbf{r}) d^3 r, \end{aligned} \tag{A13}$$

and

$$\begin{aligned} I_{\boldsymbol{V}}\{\boldsymbol{F},\boldsymbol{G}\} &= \int_D \boldsymbol{F}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & -j\omega\delta\mu\boldsymbol{\delta} \\ (\delta\sigma + j\omega\delta\epsilon)\boldsymbol{\delta} & 0 \end{bmatrix} \cdot \boldsymbol{G}(\mathbf{r}) d^3 r \\ &+ \int_D \boldsymbol{G}^T(\mathbf{r}) \cdot \begin{bmatrix} 0 & -j\omega\delta\mu\boldsymbol{\delta} \\ (j\omega\delta\epsilon - \delta\sigma)\boldsymbol{\delta} & 0 \end{bmatrix} \cdot \boldsymbol{F}(\mathbf{r}) d^3 r. \end{aligned} \tag{A14}$$

Following the same reasoning as for scalar fields, the two representation theorems for perturbations are

$$\boldsymbol{G}_S(\mathbf{r}_B,\mathbf{r}_A) - \boldsymbol{G}_S^H(\mathbf{r}_A,\mathbf{r}_B) = I_1\{\boldsymbol{G}_1(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_1(\mathbf{r},\mathbf{r}_B)\} - I_0\{\boldsymbol{G}_0(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_0(\mathbf{r},\mathbf{r}_B)\} \tag{A15}$$

and

$$\boldsymbol{G}_S(\mathbf{r}_B,\mathbf{r}_A) = I_0\{\boldsymbol{G}_S(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_0(\mathbf{r},\mathbf{r}_B)\} + \left(\boldsymbol{G}_0^H(\mathbf{r},\mathbf{r}_B)\left|\boldsymbol{V}(\mathbf{r})\right|\boldsymbol{G}_1(\mathbf{r},\mathbf{r}_A)\right). \tag{A16}$$

This leads to the same analysis of contributions to the Green's function reconstruction as in section 5 by applying the following decomposition:

$$I_0\{\boldsymbol{G}_S(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_0(\mathbf{r},\mathbf{r}_B)\} = \boldsymbol{G}_S(\mathbf{r}_B,\mathbf{r}_A) - \left(\boldsymbol{G}_0^H(\mathbf{r},\mathbf{r}_B)\left|\boldsymbol{V}(\mathbf{r})\right|\boldsymbol{G}_1(\mathbf{r},\mathbf{r}_A)\right) \tag{A17}$$

$$I_0\{\boldsymbol{G}_0(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_S(\mathbf{r},\mathbf{r}_B)\} = -\boldsymbol{G}_S^H(\mathbf{r}_A,\mathbf{r}_B) + \left(\boldsymbol{G}_0^T(\mathbf{r},\mathbf{r}_A)\left|\overline{\boldsymbol{V}}(\mathbf{r})\right|\overline{\boldsymbol{G}}_1(\mathbf{r},\mathbf{r}_B)\right) \tag{A18}$$

$$I_0\{\boldsymbol{G}_S(\mathbf{r},\mathbf{r}_A), \overline{\boldsymbol{G}}_S(\mathbf{r},\mathbf{r}_B)\} = \left(\boldsymbol{G}_S^T(\mathbf{r}_1,\mathbf{r}_A)\left|\overline{\boldsymbol{V}}(\mathbf{r}_1)\right|\overline{\boldsymbol{G}}_1(\mathbf{r}_1,\mathbf{r}_B)\right) - \left(\boldsymbol{G}_S^H(\mathbf{r}_1,\mathbf{r}_B)\left|\boldsymbol{V}(\mathbf{r}_1)\right|\boldsymbol{G}_1(\mathbf{r}_1,\mathbf{r}_A)\right). \tag{A19}$$

APPENDIX B: TREATMENT OF GENERAL UNPERTURBED BOUNDARY CONDITIONS

Here, we generalize the results of this paper to any unperturbed boundary conditions. For boundary conditions that remain unchanged after perturbing the system, both perturbed and unperturbed fields fulfill equation

$$B(\mathbf{r}) \cdot u_{0,1}(\mathbf{r}) = f(\mathbf{r}) \text{ on boundary,} \tag{B1}$$

where B is the boundary condition operator for the total volume considered D_{tot}. In particular, the unperturbed and perturbed Green's functions, $G_0(\mathbf{r},\mathbf{r}_S)$ and $G_1(\mathbf{r},\mathbf{r}_S)$, between points r and $\mathbf{r}_S$ each satisfy equation (B1). To account for this, relation (8) between unperturbed and perturbed Green's functions is modified as follows:

$$G_1(\mathbf{r},\mathbf{r}_S) = G_0(\mathbf{r},\mathbf{r}_S) + \left(G_0(\mathbf{r},\mathbf{r}_1)\left|V(\mathbf{r}_1)\right|G_1(\mathbf{r}_1,\mathbf{r}_S)\right) - \mathcal{G}(\mathbf{r}) \cdot \left(V(\mathbf{r}_1)|G_1(\mathbf{r}_1,\mathbf{r}_S)\right), \tag{B2}$$

where $\mathcal{G}$ is a solution of the homogeneous unperturbed system with boundary conditions (B1):

$$H_0(\mathbf{r}) \cdot \mathcal{G}(\mathbf{r}) = 0. \tag{B3}$$

One can verify that this new formulation satisfies boundary conditions (B1) by applying operator B to equation (B2). The perturbation of Green's function $G_S(\mathbf{r},\mathbf{r}_S)$ satisfies a different expression:

$$G_S(\mathbf{r},\mathbf{r}_S) = \left(G_0(\mathbf{r},\mathbf{r}_1)\left|V(\mathbf{r}_1)\right|G_1(\mathbf{r}_1,\mathbf{r}_S)\right) - \mathcal{G}(\mathbf{r}) \cdot \left(V(\mathbf{r}_1)|G_1(\mathbf{r}_1,\mathbf{r}_S)\right). \tag{B4}$$

The main results of this article, however, remain unchanged. We introduce the interferometric operator and derive the same general representation theorem (23) as for homogeneous boundaries. Additional derivations are needed in order to demonstrate expression (30). Consider equation (B4) for $G_S(\mathbf{r}_A,\mathbf{r}_B)$, and insert the general representation theorem for unperturbed media $G_0(\mathbf{r}_B, \mathbf{r}_1) = I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + \overline{G}_0(\mathbf{r}_1, \mathbf{r}_B)$ to obtain

$$\begin{aligned} G_S(\mathbf{r}_B, \mathbf{r}_A) &= \big(I_0\{G_0(\mathbf{r}, \mathbf{r}_1), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\}\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1, \mathbf{r}_A)\big) \\ &+ \big(\overline{G}_0(\mathbf{r}_1, \mathbf{r}_B)\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1, \mathbf{r}_A)\big) \\ &- \mathcal{G}(\mathbf{r}_B)\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A)) \\ &= I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &+ \big(\overline{G}_0(\mathbf{r}_1, \mathbf{r}_B)\,|V(\mathbf{r}_1)|\,G_1(\mathbf{r}_1, \mathbf{r}_A)\big) \\ &+ I_0\{\mathcal{G}(\mathbf{r})\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))\,, \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} \\ &- \mathcal{G}(\mathbf{r}_B)\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))\,. \end{aligned} \tag{B5}$$

Additionally,

$$\begin{aligned} I_0\{\mathcal{G}(\mathbf{r})\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))\,, \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} &= (\mathcal{G}(\mathbf{r})\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))\,|\underbrace{\overline{H}_0(\mathbf{r})|\overline{G}_0(\mathbf{r}, \mathbf{r}_B))}_{\delta(\mathbf{r}-\mathbf{r}_B)} \\ &- (\overline{G}_0(\mathbf{r}, \mathbf{r}_B)|\underbrace{H_0(\mathbf{r})|\mathcal{G}(\mathbf{r})}_{=0}\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))) \\ &= \mathcal{G}(\mathbf{r}_B)\cdot(V(\mathbf{r}_1)|G_1(\mathbf{r}_1, \mathbf{r}_A))\,. \end{aligned} \tag{B6}$$

Summing these two equations yields

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = I_0\{G_S(\mathbf{r}, \mathbf{r}_A), \overline{G}_0(\mathbf{r}, \mathbf{r}_B)\} + \big(\overline{G}_0(\mathbf{r}, \mathbf{r}_B)\,|V(\mathbf{r})|\,G_1(\mathbf{r}, \mathbf{r}_A)\big)\,. \tag{B7}$$

Equation (B7) is identical to representation theorem (30), which holds for unperturbed homogeneous boundary conditions. By analogy, one can show that all the results presented in section 5 holds for any type of unperturbed boundary conditions.

APPENDIX C: PROPERTIES OF SELF-ADJOINT DIFFERENTIAL OPERATOR: VOLUME/SURFACE INTEGRALS AND SPATIAL RECIPROCITY

The interferometric operator is defined as

$$\begin{aligned} I_H\{f, g\} &= \big(f\,|\overline{H}|\,g\big) - (g\,|H|\,f) \\ &= \int_D \big(f\cdot\overline{H}\cdot g - g\cdot H\cdot f\big)\,dV. \end{aligned} \tag{C1}$$

In section 3, we explain why for practical applications it is useful to convert a volume into a surface integral to reduce the integration over the sub-volume D to its bounding surface δD. In this appendix, we show how this relates to the concept of self-adjoint operator. We introduce what is sometimes referred to as *extended* Green's identify in the literature (Lanczos 1996) and define the adjoint $\tilde{H}$ of a linear differential operator H. The adjoint is the unique operator such that for any pair of functions (f, g), an operator P_H exits and

$$\int_D \Big(g\cdot H\cdot f - f\cdot\overline{\tilde{H}}\cdot g\Big)\,dV = -\oint_{\delta D} P_H(f, g)\cdot\hat{\mathbf{n}}dS = \text{boundary term.} \tag{C2}$$

A differential operator is self-adjoint if $H = \tilde{H}$. For self-adjoint operators, equation (C1) can be written using the *extended* Green's identify and consequently,

$$I_H\{f, g\} = \oint_{\delta D} P_H(f, g)\cdot\hat{\mathbf{n}}dS, \tag{C3}$$

so the general representation theorem becomes

$$G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\delta D} P_H(G(\mathbf{r}, \mathbf{r}_A), \overline{G}(\mathbf{r}, \mathbf{r}_B))\cdot\hat{\mathbf{n}}dS. \tag{C4}$$

For self-adjoint operators, in order to efficiently extract the Green's function between two points A and B, we need to know the operator P_H, which depends on the properties of the system, and the Green's functions on an enclosing surface δD. For more general systems ($H \neq \tilde{H}$), relation (C4) is no longer valid, but we can alway decompose the interferometric operator into surface and volume integrals and express the representation theorem as

$$\begin{aligned} G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) &= \oint_{\delta D} P_H(G(\mathbf{r}, \mathbf{r}_A), \overline{G}(\mathbf{r}, \mathbf{r}_B)) \cdot \hat{\mathbf{n}} dS \\ &+ \int_D G(\mathbf{r}, \mathbf{r}_A) \cdot \left(\overline{H - \tilde{H}}\right) \cdot G(\mathbf{r}, \mathbf{r}_B) dV. \end{aligned} \tag{C5}$$

Note that the results of this paper do not require space- and time-reciprocity. This means that the order of spatial coordinates matters in the relations we establish. To facilitate the use and interpretation of representation theorems (25) and (30) in practice, we desire systems that are spatially reciprocal, as holds for particular boundary conditions and symmetry of linear differential operators. For example, consider a representation theorem of the convolution type. By analogy with the representation theorem of the correlation type obtained in section 3, we get

$$G(\mathbf{r}_B, \mathbf{r}_A) - G(\mathbf{r}_A, \mathbf{r}_B) = \int_D \left(G(\mathbf{r}, \mathbf{r}_A) \cdot H(\mathbf{r}) \cdot G(\mathbf{r}, \mathbf{r}_B) - G(\mathbf{r}, \mathbf{r}_B) \cdot H(\mathbf{r}) \cdot G(\mathbf{r}, \mathbf{r}_A) \right) dV. \tag{C6}$$

For operators such that $\overline{H} = \tilde{H}$, which include self-adjoint real operators, e.g., the wave propagator, use of Green's identity (C2) with $D = D_{tot}$ yields

$$G(\mathbf{r}_B, \mathbf{r}_A) - G(\mathbf{r}_A, \mathbf{r}_B) = \oint_{\delta D_{tot}} P_H(G(\mathbf{r}, \mathbf{r}_A), G(\mathbf{r}, \mathbf{r}_B)) \cdot \hat{\mathbf{n}} dS. \tag{C7}$$

Depending on boundary conditions, the right-hand side of equation (C7) vanishes and consequently, we obtain $G(\mathbf{r}_B, \mathbf{r}_A) = G(\mathbf{r}_A, \mathbf{r}_B)$, i.e., spatial reciprocity. Typically, the integral $\oint_{\delta D_{tot}} P_H(G(\mathbf{r}, \mathbf{r}_A), G(\mathbf{r}, \mathbf{r}_B)) \cdot \hat{\mathbf{n}} dS$ will go to zero if the Green's functions $G(\mathbf{r}, \mathbf{r}_{A,B})$ or their derivatives vanish on δD_{tot}. For acoustic waves, the Sommerfield radiation and free surface conditions lead to spatial reciprocity. Systems with free boundaries, however, are of limited interest because we cannot practically apply interferometry. Indeed, for such systems, equation (C4) shows that $G(\mathbf{r}_B, \mathbf{r}_A) - \overline{G}(\mathbf{r}_A, \mathbf{r}_B) = 0$; that is, for self-adjoint operators, free boundaries always lead to reconstruction of a null signal.