# Geometrical spreading in a layered transversely isotropic medium with a vertical symmetry axis

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**ABSTRACT**

We present equations for the relative geometrical spreading of multiple reflected and transmitted and possibly mode-converted $P$ and $S$ waves in a heterogeneous TIV medium, and for a stack of homogeneous TIV-layers. We show that relatively simple expressions are obtained when the geometrical spreading is expressed in terms of group velocities. In weakly anisotropic media, we obtain simple expressions also in terms of phase velocities.
Our analytical equations for the geometrical spreading based on the nonhyperbolic traveltime formula of Tsvankin and Thomsen, are such that the geometrical spreading can be expressed in terms of the parameters used in time-processing of seismic data. Potential applications of this approximation are in prestack Kirchhoff time migration and AVO/AVA analysis of surface seismic and multicomponent seismic data.
Numerical examples show that the weak-anisotropy approximation to geometrical spreading compares well with ray tracing result for P-waves. It is less accurate for SV-waves, but has qualitatively the correct form. The nonhyperbolic equation for geometrical spreading compares well with ray tracing results for P-waves for offset-to-depth ratios less than five. For $SV$-waves, the analytical formula is not accurate and breaks down at offset-to-depth ratios less than unity. The numerical results are in agreement with the range of validity for the nonhyperbolic traveltime equations.

**Key words:** Geometrical spreading, TIV medium, weak anisotropy, nonhyperbolic traveltime

## 1 INTRODUCTION

Offset-dependent geometrical spreading is needed to compute the weight functions for prestack Kirchhoff migration. Moreover, seismic data must be compensated for geometrical spreading before amplitude-versus-offset (AVO) or angle (AVA) analysis can be applied to study reflection coefficients as a function of offset or incidence angle. If a velocity-depth model is available, the relative geometrical spreading can be computed by dynamic ray tracing. Numerical implementation of dynamic ray tracing, however, is cumbersome and time-consuming. For seismic time-processing methods it is useful to derive analytical expressions that can be computed without the use of numerical ray tracing. A problem of great practical interest is to express the geometrical spreading by using only measured traveltimes.

Ursin (1990) derived exact formulae for the in-plane and out-of-plane relative geometrical spreading in a horizontally layered isotropic elastic medium with the source and receivers in the same layer. He also expressed the relative geometrical spreading in terms of traveltime parameters. These formulae are routinely used in seismic time-processing. The decomposition into in-plane and out-of-plane relative geometrical spreading is important in prestack time migration (Ekren and Ursin, 1999).

Zhou and McMechan (2000) derived an analytical formula for the geometrical spreading of $P$-waves in layered transversely isotropic media with a vertical symmetry axis (TIV medium) with the source and receivers in the same layer. They did not, however, separate the geometrical spreading into in-plane and out-of-plane factors. Ettrich et al. (2000) considered the out-of-plane contribution to the geometrical spreading for wave propagation in a symmetry plane of transversely isotropic or orthorhombic media with arbitrary orientation of the other symmetry planes.

We generalize the results of Ursin (1990) to multiply reflected and converted $qP$ and $qSV$ waves in a layered TIV medium with the source and receiver possibly in different layers. This makes the results valid for surface seismic data as well as vertical seismic profile (VSP) and ocean-bottom seismic (OBS) data. The results are also valid for $SH$ waves. With the relative geometrical spreading expressed in terms of traveltime derivatives (Schleicher et al., 2001), it naturally separates into in-plane and out-of-plane factors. This leads to exact expressions in terms of the group and phase velocities. We also derive simpler approximate expressions for the geometrical spreading in weakly anisotropic media in terms of the phase velocities.

Tsvankin and Thomsen (1994) presented a nonhyperbolic traveltime formula for pure-mode $P$- and $SV$-waves and $P$-$SV$-converted waves in TIV media. For $SV$-waves, the nonhyperbolic equation becomes accurate only with numerically fitted coefficients. From the nonhyperbolic traveltime formula, we derive a closed-form equation for the relative geometrical spreading. Based on the nonhyperbolic traveltime equation, the relative geometrical spreading can be expressed in terms of the parameters commonly used in time-processing of $P$-waves and $P$-$SV$-converted waves in TIV-media.

## 2 THE GEOMETRIC RAY APPROXIMATION

Wave propagation in a heterogeneous anisotropic elastic solid is governed by the elasto-dynamic equations (Aki and Richards, 1980). In the frequency domain, the equations can be written as

$$\omega^2 \rho U_i + (c_{ijkl} U_{k,l})_{,j} = 0, \tag{1}$$

where $\omega$ is frequency, $U_i = U_i(\mathbf{x}, \omega)$ is the $i$ component of the particle displacement, $\rho = \rho(\mathbf{x})$ is the density, and $c_{ijkl} = c_{ijkl}(\mathbf{x})$ are the elastic parameters of the medium at position $\mathbf{x} = (x_i, x_2, x_3)$. The elastic Hooke's tensor satisfies the symmetry relations $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. In equation (1), the notation "$,j$" means $\partial/\partial x_j$, and repeated latin indices are summed according to Einstein's summation convention.

The Green's function $G_{in}(\mathbf{x}, \omega, \mathbf{x}^s)$ satisfies the equation

$$\omega^2 \rho G_{in} + (c_{ijkl} G_{kn,l})_{,j} = -\delta_{in} \delta(\mathbf{x} - \mathbf{x}^s). \tag{2}$$

For homogeneous boundary conditions, it also satisfies the spatial reciprocity relation (Aki and Richards, 1980)

$$G_{ij}(\mathbf{x}, \omega; \mathbf{x}^s) = G_{ji}(\mathbf{x}^s, \omega; \mathbf{x}). \tag{3}$$

For a specific ray connecting source point $\mathbf{x}^s$ to receiver point $\mathbf{x}^r$, the geometrical ray approximation (GRA) to the elasto-dynamic Green's function is (Červený, 1995; Chapman and Coates, 1994)

$$G_{ij}(\mathbf{x}^r, \omega; \mathbf{x}^s) = h_i(\mathbf{x}^s) \mathcal{A}(\mathbf{x}^r, \mathbf{x}^s) e^{i\omega T(\mathbf{X}^r, \mathbf{X}^s)} h_j(\mathbf{x}^s), \tag{4}$$

where $\mathbf{h}(\mathbf{x}^s)$ and $\mathbf{h}(\mathbf{x}^r)$ are the unit polarization vectors for the ray at the source $\mathbf{x}^s$ and the receiver $\mathbf{x}^r$, respectively. $T(\mathbf{x}^r, \mathbf{x}^s)$ is the traveltime along the ray from $\mathbf{x}^s$ to $\mathbf{x}^r$, and $\mathcal{A}(\mathbf{x}^r, \mathbf{x}^s)$ is a complex amplitude function taking into account possible caustics and phase shift at the source. For a multiple-reflected and transmitted, and possibly mode-converted, wave, the GRA has to be continued across each interface that separates the smooth parts of the elastic medium. This is done by multiplication with the proper plane-wave reflection or transmission ($R/T$) coefficient, and also with a factor (Červený, 1995)

$$\left[ \frac{(\rho V \cos\alpha)_{out}}{(\rho V \cos\alpha)_{in}} \right]^{1/2}, \tag{5}$$

where $V$ is the group velocity, and $\alpha$ is the angle that the group velocity (and thus the ray) makes with the interface normal. When this factor is included in the $R/T$ coefficient, the resulting $R/T$ coefficient is normalized with respect to the energy flux normal to the interface. With this definition, the $R/T$ coefficients satisfy reciprocity and are called *reciprocal $R/T$ coefficients* by Červený (1995). By a proper normalization of the amplitudes at the source and receiver, the remaining amplitude factors are attributable to the relative geometrical spreading, which is also reciprocal. The relative geometrical spreading may be expressed in terms of mixed second-order traveltime derivatives with respect to local coordinates in the wavefront plane, normal to the phase velocity $v(\mathbf{x})$ (Chapman and Coates, 1994). However, using mixed second-order traveltime derivatives with respect to local coordinates normal to the group velocity yields much simpler expressions for a reflected or transmitted ray. This approach is shown for a reflected and possibly mode-converted wave by Schleicher et al. (2001). The amplitude of a multiply reflected and transmitted, possibly mode-converted, wave can be written as

$$\begin{aligned} \mathcal{A}(\mathbf{x}^r, \mathbf{x}^s) \quad = \quad & \frac{e^{-i\frac{\pi}{2}\mathrm{sgn}(\omega)\kappa(\mathbf{X}^r, \mathbf{X}^s)}}{4\pi[\rho(\mathbf{x}^r)V(\mathbf{x}^r)\rho(\mathbf{x}^s)V(\mathbf{x}^s)]^{1/2}} \\ & \times \frac{1}{\mathcal{L}(\mathbf{x}^r, \mathbf{x}^s)} \prod_k \mathcal{R}_k, \end{aligned} \tag{6}$$

where $\prod_k \mathcal{R}_k$ is the product of the plane-wave $R/T$ coefficients, normalized with respect to the vertical energy flux at all interfaces crossed by the ray, and $\kappa$ is the KMAH index, taking into account possible caustics. The relative geometrical spreading factor is

$$\mathcal{L}(\mathbf{x}^r, \mathbf{x}^s) = |\det \mathbf{Y}(\mathbf{x}^r, \mathbf{x}^s)|^{1/2}, \tag{7}$$

where the matrix $\mathbf{Y}$ can be computed from

$$Y_{ij}^{-1} = -\frac{\partial^2 T(\mathbf{x}^r, \mathbf{x}^s)}{\partial g_i^s \partial g_j^r}, \quad i, j = 1, 2, \tag{8}$$

and $g_i^s$ and $g_j^r$ are local coordinates in the planes normal to the ray (and group velocity), respectively. Schleicher et al. (2001) showed that

$$\frac{\partial^2 T}{\partial g_3^s \partial g_j^r} = \frac{\partial^2 T}{\partial g_i^s \partial g_3^r} = 0, \tag{9}$$

where both $g_3^s$ and $g_3^r$ point in the direction of the ray. This is so because

$$\frac{\partial T}{\partial g_3^s} = \frac{1}{V(\mathbf{x}^s)}, \tag{10}$$

and the derivative is taken along the ray for which $V(\mathbf{x}^s)$ is constant.

To express the relative geometrical spreading in terms of the general Cartesian coordinate system, we let the rays at the source and receiver be in the $(x_1, x_3)$-plane. Then, the change from local ray coordinates to global coordinates at the source and receiver are rotations around the $x_2$-axis. Taking into account equation (9), gives

$$\begin{aligned} \mathbf{B} &= -\begin{bmatrix} \frac{\partial^2 T}{\partial x_1^s \partial x_1^r} & \frac{\partial^2 T}{\partial x_1^s \partial x_2^r} \\ \frac{\partial^2 T}{\partial x_2^s \partial x_1^r} & \frac{\partial^2 T}{\partial x_2^s \partial x_2^r} \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha^r & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Y}^{-1} \begin{bmatrix} \cos\alpha^s & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{11}$$

where $\alpha^s$ and $\alpha^r$ are the angles the ray makes with the $x_3$-axis at the receiver and source, respectively. Equation (11) gives the relative geometrical spreading expressed by standard mixed traveltime derivatives as

$$\mathcal{L}(\mathbf{x}^r, \mathbf{x}^s) = [\cos\alpha^r \cos\alpha^s]^{1/2} |\det \mathbf{B}|^{-1/2}. \tag{12}$$

## 3 RELATIVE GEOMETRICAL PREADING IN A VERTICALLY HETEROGENEOUS TIV MEDIUM

Consider wave propagation in a transversely isotropic medium with vertical symmetry axis (TIV medium), where the elastic parameters are functions of depth only (1-D TIV medium). For a multiple reflected $SH$-wave or a multiple reflected and converted $qP$-$qSV$-wave, the GRA amplitude is given by equation (6). From the rotation and translation symmetry of the TIV-medium, it follows that, for fixed source and receiver depths, the traveltime is a function of only the distance $r$ between the source and receiver. As shown in Appendix A, the matrix of mixed traveltime derivatives is diagonal, and the relative geometrical spreading, given in equation (12), separates into in-plane and out-of-plane factors

$$\mathcal{L}(\mathbf{x}^r, \mathbf{x}^s) = \mathcal{L}^{\parallel}(\mathbf{x}^r, \mathbf{x}^s) \mathcal{L}^{\perp}(\mathbf{x}^r, \mathbf{x}^s), \tag{13}$$

where the in-plane factor is

$$\mathcal{L}^{\parallel}(\mathbf{x}^r, \mathbf{x}^s) = \left[\cos\alpha^r \cos\alpha^s \left(\frac{\partial^2 T}{\partial r^2}\right)^{-1}\right]^{1/2}, \tag{14}$$

and the out-of-plane factor is

$$\mathcal{L}^{\perp}(\mathbf{x}^r, \mathbf{x}^s) = \left[\frac{1}{r}\frac{\partial T}{\partial r}\right]^{-1/2}. \tag{15}$$

The horizontal slowness

$$p = \frac{\partial T}{\partial r} = \frac{\sin\theta}{v}, \tag{16}$$

is constant along the ray. Here, $v$ is the phase velocity, and $\theta$ is the angle the phase velocity makes with the vertical axis. The relative geometrical spreading can be expressed in terms of the horizontal slowness as

$$\mathcal{L}^{\parallel}(r) = \left[\cos\alpha^r \cos\alpha^s \frac{\partial r}{\partial p}\right]^{1/2}, \tag{17}$$

and

$$\mathcal{L}^{\perp}(r) = \left[\frac{r}{p}\right]^{1/2}. \tag{18}$$

## 4 STACK OF HORIZONTAL HOMOGENEOUS TIV LAYERS

For a stack of horizontal homogeneous TIV layers, we obtain

$$\mathcal{L}^{\perp} = \left[\frac{r}{p}\right] = \left[\frac{1}{p}\sum_k \Delta z_k \tan\alpha_k\right]^{1/2}, \tag{19}$$

where $\Delta z_k$ is layer thickness and $\alpha_k$ is the group angle in layer $k$. The sum is taken over all layers through which the wave propagates. Each time the wave passes through a new layer, a new term is added with the proper group angle, depending on the wave mode. Substitution for the horizontal slowness gives

$$\mathcal{L}^{\perp} = \left[\sum_k \frac{v_k \Delta z_k \tan\alpha_k}{\sin\theta_k}\right]^{1/2}. \tag{20}$$

Using results from Appendix B, we obtain

$$\mathcal{L}^{\perp} = \left[\sum_k \frac{V_k \Delta z_k}{\cos\alpha_k - \sin\alpha_k \tan(\alpha_k - \theta_k)}\right]^{1/2}, \tag{21}$$

in terms of the group velocities.

For the in-plane geometrical spreading, we obtain

$$\mathcal{L}^{\parallel} = \left[\cos\alpha^r \cos\alpha^s \sum_k \frac{\Delta z_k}{\cos^2\alpha_k} \frac{d\alpha_k}{d\theta_k} \frac{d\theta_k}{dp}\right]^{1/2}, \tag{22}$$

From equation (16), we derive

$$\frac{dp}{d\theta} = \frac{\cos\theta}{v}\left[1 - \tan\theta\frac{1}{v}\frac{dv}{d\theta}\right] = \frac{V}{v^2}\cos\alpha. \quad (23)$$

Combining these equations with results from Appendix B, the in-plane geometrical spreading, in terms of group velocities, can be written as

$$\mathcal{L}^{\parallel} = \left[\cos\alpha^r\cos\alpha^s \times \sum_k \frac{V_k\Delta z_k}{\cos^3\alpha_k\left(1 + 2\tan^2(\alpha_k - \theta_k) - \frac{1}{V_k}\frac{d^2V_k}{d\alpha_k^2}\right)}\right]^{1/2} \quad (24)$$

In terms of the phase velocities, we obtain

$$\mathcal{L}^{\parallel} = \left[\cos\alpha^r\cos\alpha^s \times \sum_k \frac{v_k\Delta z_k\left(1 + \frac{1}{v_k}\frac{d^2v_k}{d\theta_k^2}\right)}{\cos^3\theta_k\left(1 - \tan\theta_k\tan(\alpha_k - \theta_k)\right)^3}\right]^{1/2} \quad (25)$$

When the layers are all isotropic, $\theta = \alpha =$ constant, and the formulas reduce to the expressions presented by Ursin (1990)

$$\mathcal{L}^{\perp} = \left[\sum_k \frac{v_k\Delta z_k}{\cos\theta_k}\right]^{1/2}, \quad (26)$$

$$\mathcal{L}^{\parallel} = \left[\cos\theta^r\cos\theta^s\sum_k\frac{v_k\Delta z_k}{\cos^3\theta_k}\right]^{1/2}. \quad (27)$$

## 5 WEAK ANISOTROPY APPROXIMATIONS

For weak anisotropy, we use the approximate phase velocities given by Thomsen (1996), and slightly modified by Stovas and Ursin (2002). For $P$-waves, we need

$$v_P(\theta) \simeq \alpha_0[1 + \delta\sin^2\theta\cos^2\theta + \epsilon\sin^4\theta], \quad (28)$$

$$\frac{1}{v_P}\frac{dv_P}{d\theta} \simeq \sin\theta\cos\theta f_P(\theta), \quad (29)$$

$$f_P(\theta) = 2\delta\cos 2\theta + 4\epsilon\sin^2\theta, \quad (30)$$

$$\frac{1}{v_P}\frac{d^2v_P}{d\theta^2} \simeq 2[\delta + (6\epsilon - 8\delta)\sin^2\theta - 8(\epsilon - \delta)\sin^4\theta]. \quad (31)$$

For $SV$ waves, we need

$$v_{SV}(\theta) \simeq \beta_0\left[1 + \frac{\alpha_0^2}{\beta_0^2}\eta\sin^2\theta\cos^2\theta\right], \quad (32)$$

$$\frac{1}{v_{SV}}\frac{dv_{SV}}{d\theta} \simeq \sin\theta\cos\theta f_{SV}(\theta), \quad (33)$$

$$f_{SV}(\theta) = 2\frac{\alpha_0^2}{\beta_0^2}\eta\cos 2\theta, \quad (34)$$

$$\frac{1}{v_{SV}}\frac{d^2v_{SV}}{d\theta^2} \simeq 2\frac{\alpha_0^2}{\beta_0^2}\eta(1 - 2\sin^2 2\theta), \quad (35)$$

and for $SH$-waves

$$v_{SH}(\theta) \simeq \beta_0\left(1 + \gamma\sin^2\theta\right), \quad (36)$$

$$\frac{1}{v_{SH}}\frac{dv_{SH}}{d\theta} \simeq \sin\theta\cos\theta f_{SH}(\theta), \quad (37)$$

$$f_{SH}(\theta) = 2\gamma, \quad (38)$$

$$\frac{1}{v_{SH}}\frac{d^2v_{SH}}{d\theta^2} \simeq 2\gamma\cos^2\theta. \quad (39)$$

In the formulas above, the vertical velocities and anisotropy parameters are given by (Thomsen, 1986; Alkalifah, 1997)

$$\alpha_0^2 = \frac{C_{33}}{\rho}, \quad \beta_0^2 = \frac{C_{44}}{\rho}, \quad (40)$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}, \quad (41)$$

$$\epsilon = \frac{C_{11} - C_{33}}{2C_{33}}, \quad (42)$$

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta} \simeq \epsilon - \delta, \quad (43)$$

$$\gamma = \frac{C_{66} - C_{44}}{2C_{44}}, \quad (44)$$

where $\alpha_0$ and $\beta_0$ are the vertical $P$- and $S$-wave velocities, respectively. From the elastodynamic equations, Stovas and Ursin (2001) derived

$$\delta_M = \frac{C_{13} + 2C_{44} - C_{33}}{2C_{33}}, \quad (45)$$

$$\eta_M = \frac{C_{11}C_{33} - (C_{13} + 2C_{44})^2}{2C_{33}^2}, \quad (46)$$

which are identical to Thomsen's definition to lowest order in anisotropy. $\delta_M$ was also proposed by Sayers (1995).

For weak anisotropy, we note that from equation (B8) we obtain

$$\tan\alpha \simeq \tan\theta\left[1 + \left(\frac{1}{\tan\theta} + \tan\theta\right)\frac{1}{v}\frac{dv}{d\theta}\right] = \tan\theta[1 + f(\theta)], \quad (47)$$

where $f(\theta)$ is defined in equations (30), (34) and (38) above for the different wave modes. From equation (20), we now obtain a weak-anisotropy approximation for the out-of-plane relative geometrical spreading

$$\mathcal{L}^{\perp} = \left\{\sum_k \frac{v_k(\theta_k)\Delta z_k}{\cos\theta_k}[1 + f_k(\theta_k)]\right\}^{1/2}. \quad (48)$$

From equation (B3) or Thomsen (1986), we have that $v(\theta) \simeq V(\alpha)$. Then the in-plane geometrical spreading given in equation (25) can be approximated by

$$\mathcal{L}^{\parallel} = \left\{\cos\theta^r[1 - \sin^2\theta^r f^r(\theta^r)] \times \cos\theta^s[1 - \sin^2\theta^s f^s(\theta^s)] \times \sum_k \frac{v_k(\theta_k)\Delta z_k\left(1 + \frac{1}{v_k}\frac{d^2v_k}{d\theta_k^2}\right)}{\cos^3\theta_k\left[1 - \sin^2\theta_k f_k(\theta_k)\right]^3}\right\}^{1/2}, \quad (49)$$

where we have used equation (B5). In the two previous expressions, equations (30), (34) and (38) have to be used, depending on the wave mode in a specific layer.

## 6 GEOMETRICAL SPREADING FROM NONHYPERBOLIC TRAVELTIME

Tsvankin and Thomsen (1994) have shown that the nonhyperbolic traveltime $T$ in a TIV medium is given approximately by the formula

$$T^2(x) = T_0^2 + \frac{x^2}{V_{NMO}^2} + \frac{A_4 x^4}{1 + A_5 x^2}, \tag{50}$$

where $x$ is the source-receiver offset. The offset traveltime $T(x)$ and zero-offset traveltime $T_0$ can be interpreted either as two-way reflection times or as one-way transmission times (e.g. measured in a borehole). The latter is convenient when the nonhyperbolic traveltime equation is applied in prestack time migration (Hokstad and Sollie, 1999). Equation (50) can be applied for pure-mode $P$- and $S$-waves and $P$-$SV$-converted waves, provided the proper coefficients $V_{NMO}$, $A_4$ and $A_5$ are used. The NMO velocities for $P$- and $S$-waves are given by

$$V_{P,NMO}^2 = \alpha_0^2 [1 + 2\delta], \tag{51}$$

$$V_{SV,NMO}^2 = \beta_0^2 [1 + 2\sigma], \tag{52}$$

$$\sigma = \left(\frac{\alpha_0}{\beta_0}\right)^2 (\epsilon - \delta), \tag{53}$$

$$V_{SH,NMO}^2 = \beta_0^2 [1 + 2\gamma], \tag{54}$$

where $\alpha_0$ and $\beta_0$ are the vertical $P$- and $S$-wave velocities, and $\delta$, $\epsilon$ and $\gamma$ are defined in equations (41) to (44) above. The coefficients $A_4$ and $A_5$ are given by Tsvankin and Thomsen (1994). For $P$-waves, the nonhyperbolic traveltime equation can be rewritten as (Alkalifah, 1997; Grechka and Tsvankin, 1998)

$$T^2(x) = T_0^2 + \frac{x^2}{V_{P,NMO}^2} - \frac{2\eta x^4}{V_{P,NMO}^2 [T_0^2 V_{P,NMO}^2 + (1 + 2\eta) x^2]}. \tag{55}$$

The two parameters $V_{P,NMO}$ and $\eta$ are sufficient to perform time-processing (NMO, DMO, prestack time migration) of $P$-waves.

The derivatives of the traveltime function can be written as

$$\frac{dT}{dx} = \frac{H(x)}{V_{NMO}^2 T(x)} x, \tag{56}$$

$$\frac{d^2T}{dx^2} = \frac{H(x)}{V_{NMO}^2 T(x)} \times \left[1 + \frac{1}{H(x)} \frac{dH}{dx} x - \frac{H(x)}{V_{NMO}^2 T^2(x)} x^2\right], \tag{57}$$

| | $V_{P,NMO}$ (m/s) | $V_{S,NMO}$ (m/s) | $\epsilon$ | $\delta$ | $\sigma$ |
|---|---|---|---|---|---|
| $\epsilon = \delta$ | 2191 | 1000 | 0.10 | 0.10 | 0.00 |
| $\epsilon > \delta$ | 2098 | 1183 | 0.10 | 0.05 | 0.20 |
| $\epsilon < \delta$ | 2280 | 775 | 0.10 | 0.15 | -0.20 |
| Isotropy | 2000 | 1000 | 0.00 | 0.00 | 0.00 |

**Table 1.** $P$- and $S$-wave NMO velocities and Thomsen parameters for three qualitatively different cases of TIV anisotropy: $\epsilon = \delta$, $\epsilon > \delta$ and $\epsilon < \delta$, and the isotropic limit.

where we have introduced

$$H(x) = 1 + V_{NMO}^2 \frac{A_4 x^2}{1 + A_5 x^2} \left[2 - \frac{A_5 x^2}{1 + A_5 x^2}\right], \tag{58}$$

$$x\frac{dH}{dx} = 4V_{NMO}^2 \frac{A_4 x^2}{1 + A_5 x^2} \left[1 - \frac{A_5 x^2}{1 + A_5 x^2}\right]^2. \tag{59}$$

Substituting equations (56) and (57) in equations (14) and (15), we obtain the separate contributions from out-of-plane and in-plane geometrical spreading

$$\mathcal{L}^{\perp}(x) = \left[\frac{V_{NMO}^2 T(x)}{H(x)}\right]^{1/2}, \tag{60}$$

$$\mathcal{L}^{\parallel}(x) = [\cos\alpha^s \cos\alpha^r \times \frac{V_{NMO}^2 T(x)}{H(x)\left(1 + \frac{x}{H(x)}\frac{dH}{dx} - \frac{H(x)x^2}{V_{NMO}^2 T^2(x)}\right)}\Bigg]^{1/2} \tag{61}$$

Equations (60) and (61) are numerically well-behaved for all offsets, including $x = 0$. In Appendix C, we show how the factors $\cos\alpha^s$ and $\cos\alpha^r$ can be computed from the medium parameters at the source and receiver and the horizontal slowness given in equation (56). For sources and receivers in an isotropic medium, e.g., marine seismic data, we have

$$\cos\alpha = \sqrt{1 - p^2 v^2} = \cos\theta, \tag{62}$$

where $v$ is the propagation velocity.

## 7 NUMERICAL EXAMPLES

In the first numerical example, we study the nonhyperbolic approximation to the relative geometrical spreading, equations (60) and (61). The model is a homogeneous TIV medium with vertical velocities $\alpha_0 = 2000$ m/s, $\beta_0 = 1000$ m/s, and zero-offset traveltimes $T_{P0} = 1.0$ s, $T_{S0} = 2.0$ s. Keeping vertical velocities and zero offset travel times fixed, we compare three qualitatively different cases of anisotropy: $\epsilon = \delta$ (elliptic), $\epsilon > \delta$ and $\epsilon < \delta$. For comparison, we also included the isotropic case, $\epsilon = \delta = 0$. The NMO velocities for $P$- and $SV$-waves and Thomsen parameters are given in Table 1. Figures 1 and 2 show the nonhyperbolic traveltimes and inverse relative geometrical spreading $1/\mathcal{L}$ for $P$- and

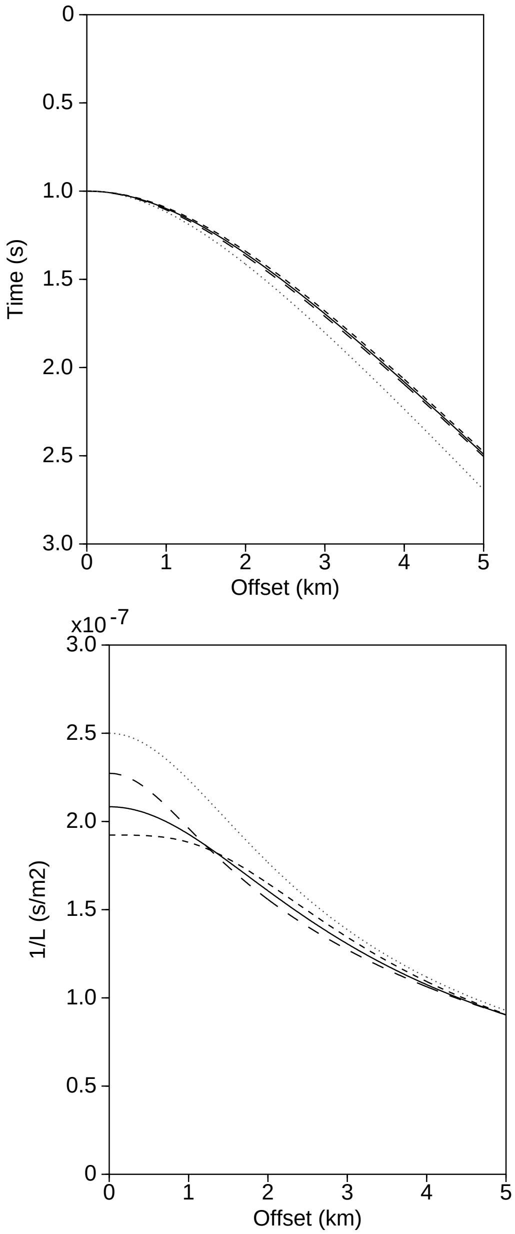


**Figure 1.** $P$-wave traveltime (top) and inverse relative geometrical spreading (bottom) for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes), and isotropic (dotted line).

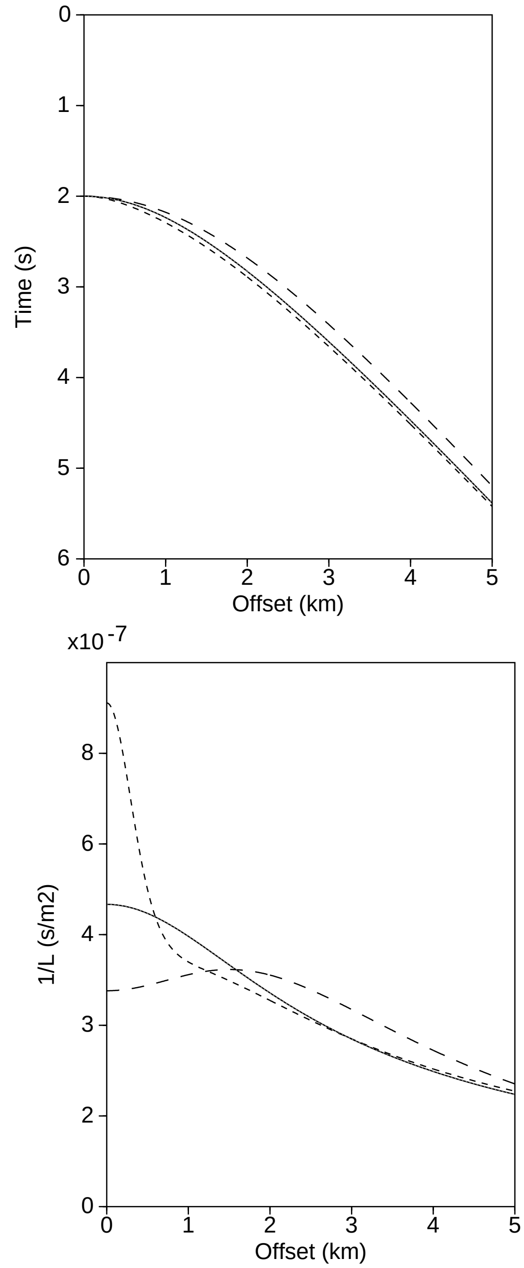


**Figure 2.** $SV$-wave traveltime (top) and inverse relative geometrical spreading (bottom) for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes), and isotropic (coincident with $\epsilon = \delta$).

$SV$-waves, respectively. The traveltimes are computed by equation (50) for offsets from 0 to 5 km. The traveltimes have been computed for all the four different models in Table 1, and can be interpreted as either two-way reflection times or as one-way transmission times. The corresponding relative geometrical spreading was computed using equations (60) and (61).

For $P$-waves, the anisotropic models give geometrical spreading that differs from that for the isotropic case for all offsets. At zero offset, anisotropy gives larger geometrical spreading whenever $\delta > 0$, which is the case for all the examples shown here. This results from the dependence of the NMO velocity on $\delta$; see equation (51). At large offsets the geometrical spreading for the three anisotropic examples approaches asymptotically the same value because all anisotropic models used have the same horizontal $P$-wave velocity $\alpha_h = \alpha_0\sqrt{1+2\epsilon}$.

| Layer | $\Delta z$ (m) | $\alpha_0$ (m/s) | $\beta_0$ (m/s) | $\delta$ | $\epsilon$ | $\sigma$ |
|---|---|---|---|---|---|---|
| 1 | 250 | 1740.0 | 390.0 | 0.05 | 0.08 | 0.60 |
| 2 | 150 | 1850.0 | 620.0 | 0.10 | 0.14 | 0.36 |
| 3 | 100 | 1940.0 | 780.0 | 0.03 | 0.10 | 0.43 |
| 4 | 160 | 2140.0 | 860.0 | -0.02 | 0.14 | 0.99 |
| 5 | 90 | 2220.0 | 890.0 | -0.05 | 0.10 | 0.93 |
| 6 | 40 | 2000.0 | 1000.0 | 0.10 | 0.14 | 0.16 |
| 7 | 100 | 1990.0 | 990.0 | 0.05 | 0.10 | 0.20 |
| 8 | 190 | 1900.0 | 950.0 | 0.04 | 0.12 | 0.32 |
| 9 | 270 | 2200.0 | 1150.0 | 0.06 | 0.18 | 0.44 |
| 10 | 170 | 2050.0 | 1130.0 | 0.10 | 0.20 | 0.33 |
| 11 | 310 | 2650.0 | 1500.0 | 0.07 | 0.10 | 0.09 |
| 12 | 300 | 2750.0 | 1530.0 | 0.10 | 0.14 | 0.13 |
| 13 | 110 | 2640.0 | 1490.0 | 0.04 | 0.08 | 0.13 |

**Table 2.** Layer thicknesses $\Delta z$, vertical velocities, and Thomsen parameters for a TIV medium with 13 layers.

For $SV$-waves, anisotropy depends mainly on the parameter $\sigma$, defined in equation (53). Therefore elliptic anisotropy $\epsilon = \delta$ is coincident with the isotropic case. The geometrical spreading at zero offset is determined by the NMO velocity in equation (52). Depending on the sign of $\sigma$, the zero-offset relative geometrical spreading is larger when $\epsilon > \delta$, or smaller when $\epsilon < \delta$, compared to the isotropic case. When anisotropy is anelliptic ($\epsilon \neq \delta$), the geometrical spreading for $SV$-waves is qualitatively very different from that of the isotropic case (Tsvankin, 1995).

For comparison, Figure 3 shows traveltimes and inverse geometrical spreading $1/\mathcal{L}$ computed from the nonhyperbolic traveltime formula, together with traveltimes and geometrical spreading from ray theory [equations (17) and (18)], for the anelliptic case with $\epsilon > \delta$. For $P$-waves, the nonhyperbolic traveltime results compare well with ray theory for all offsets shown. For $SV$-waves, we find a significant deviation from ray theory at offset-to-depth ratios larger than unity. The maximum of the inverse relative geometrical spreading is at nonzero offset when $\epsilon > \delta$. In this case the $SV$ phase velocity has a maximum at 45°. Consequently, there is a concentration of rays in this direction, which accounts for the larger amplitude and smaller geometrical spreading. The behavior of $SV$-waves in the vicinity of the velocity maximum is too complex to be modeled accurately by the nonhyperbolic traveltime equation (50). In practice, the nonhyperbolic approximation breaks down for offset-to-depth ratios less than unity.

In the second numerical example, we consider a model with 13 transversely isotropic layers. The parameters of the model are given in Table 2. Reflection traveltimes were computed using ray tracing and the nonhyperbolic traveltime equation. Geometrical spreading was computed by means of ray tracing, the weak anisotropy approximation in equations (48) and (49),

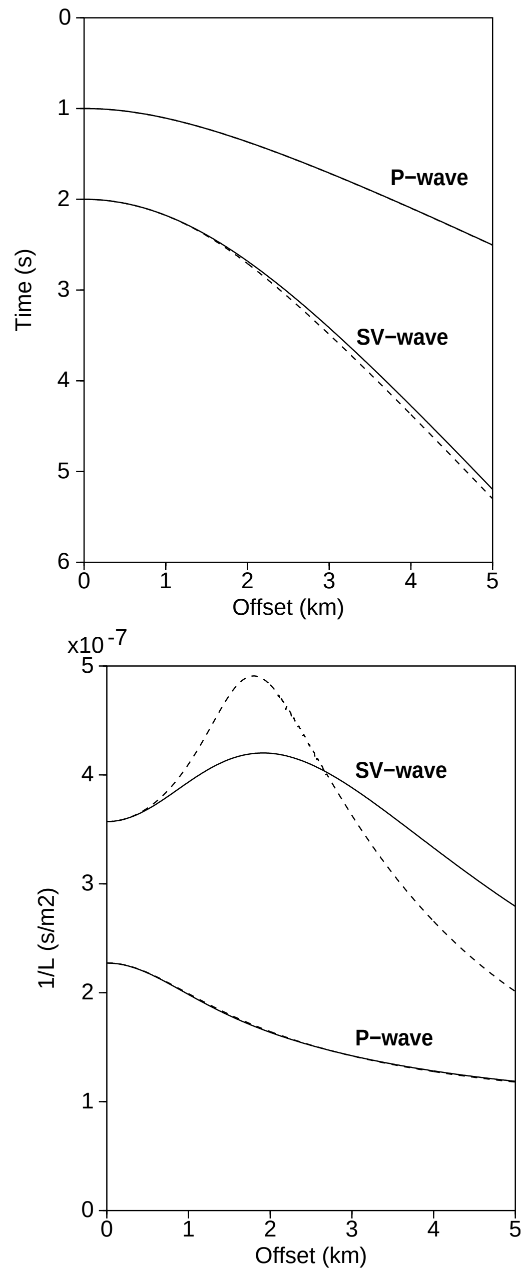


**Figure 3.** Nonhyperbolic traveltime and inverse relative geometrical spreading (solid line) compared with ray tracing results (dashed line) for the anelliptic model with $\epsilon > \delta$ in Table 1.

and the nonhyperbolic traveltime approximation in equations (60) and (61), with the source and receivers located at the surface. Figure 4 shows the two-way $PP$ traveltime and geometrical spreading from the bottom of layer 3 at 500 m depth, layer 10 at 1520 m depth and layer 12 at 2130 m depth. The weak-anisotropy approximation does not deviate significantly from the ray tracing results. The nonhyperbolic approximation for geometrical spreading is inaccurate for offset-to-depth ratios larger than five. Figure 5 shows the two-way $SV$-$SV$ traveltime and geometrical spreading from the bottom of layers 3, 10 and 12. In this case, although the weak-anisotropy approximation deviates significantly from the ray tracing results, the curve has qualitatively the correct shape. The nonhyperbolic traveltime approximation becomes inaccurate for offset-to-depth ratios less than unity.

## 8 DISCUSSION AND CONCLUSIONS

We have derived equations for the reciprocal relative geometrical spreading in a homogeneous TIV medium and for a stack of horizontal TIV-layers. The equations naturally split in separate factors for the out-of-plane and in-plane contributions, where in-plane refers to the plane defined by the source and receiver positions and the vertical. The relative geometrical spreading can be written in terms of both group and phase variables. We show that the simplest and most compact expressions are obtained in terms of group angles and group velocities. In weakly anisotropic media, we obtained simplified expressions for the geometrical spreading in terms of phase angles and phase velocities, using approximate relations between group and phase variables.

Based on the results mentioned above and the nonhyperbolic traveltime formula of Tsvankin and Thomsen (1994), we derived analytical equations for geometrical spreading. In the nonhyperbolic approximation, the geometrical spreading can be expressed in terms of the processing parameters used in time-processing of $P$-waves and $P$-$SV$-converted waves in TIV media. Potential applications of the nonhyperbolic approximation are in prestack Kirchhoff time migration and AVO/AVA analysis of surface seismic and multicomponent seismic data.

Numerical examples show that the weak-anisotropy approximation to geometrical spreading compares well with ray tracing result for P-waves. It is less accurate for SV-waves, but has qualitatively the correct form..

The nonhyperbolic equation for geometrical spreading compares well with ray tracing results for P-waves for offset-to-depth ratios less than five. For $SV$-waves, the analytical formula breaks down at offset-to-depth ratios less than unity. The numerical results are in agreement with the regime of validity for the nonhyperbolic traveltime equation. For $P$-$SV$-converted waves, the reflected $SV$ ray is usually much closer to the vertical than is the incident $P$ ray, due to Snell's law. In most practical situations, we therefore stay within the offset-to-depth range where the nonhyperbolic approximation can be applied.

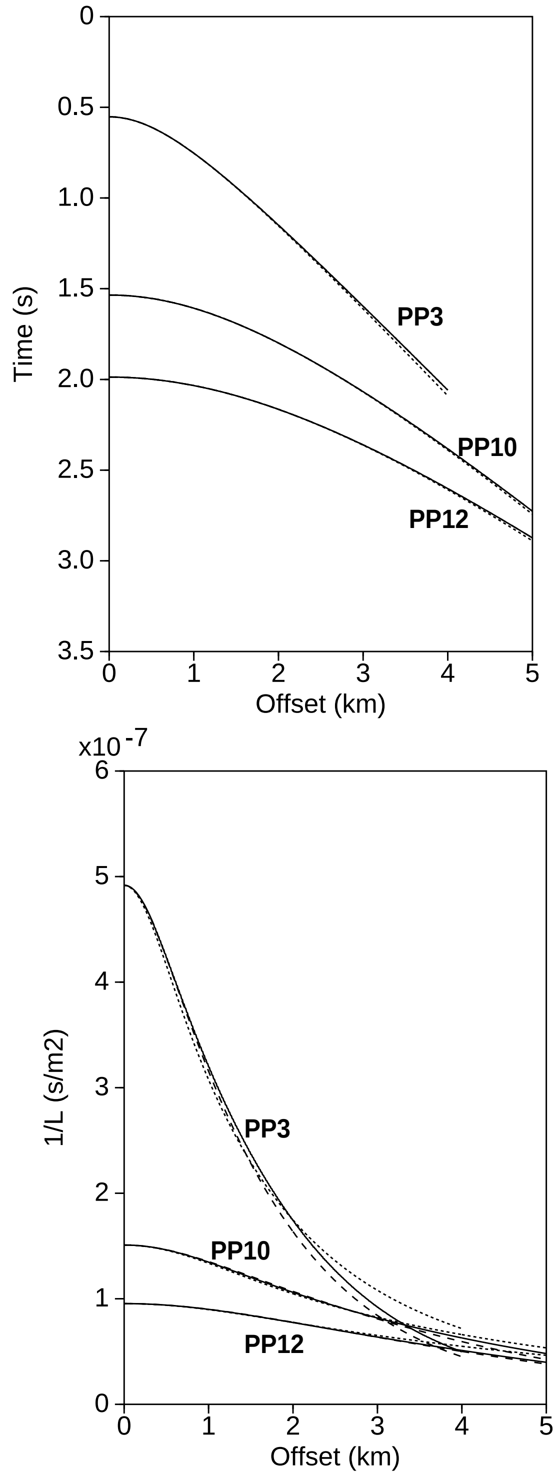


**Figure 4.** Traveltime and inverse geometrical spreading computed by ray tracing (solid line) the weak anisotropy approximation (dashed line) and nonhyperbolic traveltime equations (dotted line). $PP$-reflections from the bottom of layers 3, 10 and 12 in the TIV model given in Table 2.

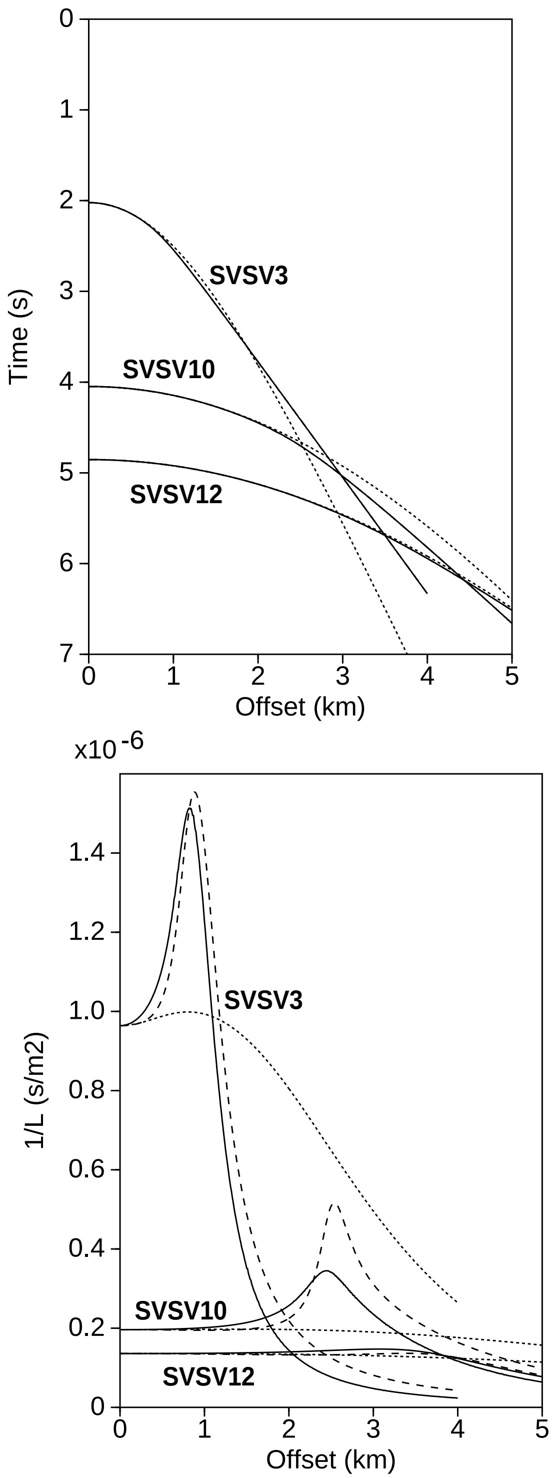


**Figure 5.** Traveltime and inverse geometrical spreading computed by ray tracing (solid line) the weak anisotropy approximation (dashed line) and nonhyperbolic traveltime equations (dotted line). $SV$-$SV$-reflections from the bottom of layers 3, 10 and 12 in the TIV model given in Table 2.

## 9 ACKNOWLEDGMENTS

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## APPENDIX A: MIXED SECOND-ORDER TRAVELTIME DERIVATIVES FOR A TIV MEDIUM

We consider a 1-D TIV medium with sources and receivers at constant, but generally different, depth $x_3$. We choose the source-receiver line to be in the $x_1$ direction. Due to the radial symmetry of the medium, traveltime $T$ depends only on the horizontal radial distance

$$r = [(x_1^s - x_1^r)^2 + (x_2^s - x_2^r)^2]^{1/2}. \tag{A1}$$

Then, we have

$$\frac{\partial T}{\partial x_i^s} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_i^s}, \quad i = 1,2, \tag{A2}$$

$$\frac{\partial T}{\partial x_i^r} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_i^r}, \quad i = 1,2, \tag{A3}$$

$$\frac{\partial^2 T}{\partial x_i^s \partial x_j^r} = \frac{\partial^2 T}{\partial r^2}\frac{\partial T}{\partial x_i^s}\frac{\partial T}{\partial x_j^r} + \frac{\partial T}{\partial r}\frac{\partial^2 T}{\partial x_i^s \partial x_j^r}, i,j = 1,2. \tag{A4}$$

Furthermore

$$\frac{\partial r}{\partial x_i^s} = \frac{x_i^s - x_i^r}{r}, \tag{A5}$$

$$\frac{\partial r}{\partial x_i^r} = -\frac{x_i^s - x_i^r}{r}, \tag{A6}$$

and

$$\frac{\partial^2 r}{\partial x_i^s \partial x_j^r} = -\frac{1}{r}\delta_{ij} + \frac{(x_i^s - x_i^r)(x_j^s - x_j^r)}{r^3}, \tag{A7}$$

where $\delta_{ij}$ is the Kronecker delta. In the general source-receiver direction $x_2^r = x_2^s = 0$ and $r = x_1^s - x_1^r$, assuming $x_1^s > x_1^r$. Then, from equations (A5) to (A7), we obtain

$$\frac{\partial r}{\partial x_2^s} = \frac{\partial r}{\partial x_2^r} = 0, \tag{A8}$$

$$\frac{\partial r}{\partial x_1^s} = -\frac{\partial r}{\partial x_2^r} = 1, \tag{A9}$$

and

$$\frac{\partial^2 r}{\partial x_1^s \partial x_1^r} = \frac{\partial^2 r}{\partial x_1^s \partial x_2^r} = \frac{\partial^2 r}{\partial x_2^s \partial x_1^r} = 0, \tag{A10}$$

$$\frac{\partial^2 r}{\partial x_2^s \partial x_2^r} = -\frac{1}{r}. \tag{A11}$$

When these equations are used in equation (A4), the final results are

$$\frac{\partial^2 T}{\partial x_1^s \partial x_1^r} = -\frac{\partial^2 T}{\partial r^2}, \tag{A12}$$

$$\frac{\partial^2 T}{\partial x_1^s \partial x_2^r} = \frac{\partial^2 T}{\partial x_2^s \partial x_1^r} = 0, \tag{A13}$$

$$\frac{\partial^2 T}{\partial x_2^s \partial x_2^r} = -\frac{1}{r}\frac{\partial T}{\partial r}. \tag{A14}$$

## APPENDIX B: RELATIONS BETWEEN GROUP AND PHASE VELOCITIES AND ANGLES FOR ANISOTROPIC MEDIA

For anisotropic media we always have (Červený, 1995)

$$\cos(\alpha - \theta) = \frac{v}{V}, \tag{B1}$$

where $v$ is the phase velocity, $V$ is the group velocity, and $\theta$ and $\alpha$ are the phase and group angles, respectively. Taking the derivatives of this equation with respect to $\theta$, gives

$$\frac{1}{v}\frac{dv}{d\theta} - \tan(\alpha - \theta) = \left[\frac{1}{V}\frac{dV}{d\alpha} - \tan(\alpha - \theta)\right]\frac{d\alpha}{d\theta}. \tag{B2}$$

Berryman (1979) showed that in a TIV medium

$$V^2 = v^2 + \left(\frac{dv}{d\theta}\right)^2, \tag{B3}$$

from which follows that

$$\left(\frac{1}{v}\frac{dv}{d\theta}\right)^2 = \frac{1}{\cos^2(\alpha - \theta)} - 1 = \tan^2(\alpha - \theta). \tag{B4}$$

From this and equation (B2), we obtain

$$\frac{1}{v}\frac{dv}{d\theta} = \frac{1}{V}\frac{dV}{d\alpha} = \tan(\alpha - \theta). \tag{B5}$$

Using equations (B1) and (B5), and standard trigonometrical relations give

$$\begin{aligned} \cos\alpha &= \cos\theta\cos(\alpha - \theta) - \sin\theta\sin(\alpha - \theta) \\ &= \cos\theta\cos(\alpha - \theta)[1 - \tan\theta\tan(\alpha - \theta)] \\ &= \cos\theta\frac{v}{V}\left[1 - \tan\theta\frac{1}{v}\frac{dv}{d\theta}\right], \end{aligned} \tag{B6}$$

$$\begin{aligned} \sin\alpha &= \sin\theta\cos(\alpha - \theta) + \cos\theta\sin(\alpha - \theta) \\ &= \sin\theta\cos(\alpha - \theta)\left[1 + \frac{1}{\tan\theta}\tan(\alpha - \theta)\right] \\ &= \sin\theta\frac{v}{V}\left[1 + \frac{1}{\tan\theta}\frac{1}{v}\frac{dv}{d\theta}\right], \end{aligned} \tag{B7}$$

Combining the last two equations, gives

$$\tan\alpha = \tan\theta\frac{1 + \frac{1}{\tan\theta}\frac{1}{v}\frac{dv}{d\theta}}{1 - \tan\theta\frac{1}{v}\frac{dv}{d\theta}}. \tag{B8}$$

In the same way we obtain

$$\begin{aligned} \cos\theta &= \cos\alpha\cos(\theta - \alpha) - \sin\alpha\sin(\theta - \alpha) \\ &= \cos\alpha\cos(\theta - \alpha)[1 - \tan\alpha\tan(\theta - \alpha)] \\ &= \cos\alpha\frac{v}{V}\left[1 + \tan\alpha\frac{1}{V}\frac{dV}{d\alpha}\right], \end{aligned} \tag{B9}$$

$$\begin{aligned} \sin\theta &= \sin\alpha\cos(\theta - \alpha) + \cos\alpha\sin(\theta - \alpha) \\ &= \sin\alpha\cos(\theta - \alpha)\left[1 + \frac{1}{\tan\alpha}\tan(\theta - \alpha)\right] \\ &= \sin\alpha\frac{v}{V}\left[1 - \frac{1}{\tan\alpha}\frac{1}{V}\frac{dV}{d\alpha}\right], \end{aligned} \tag{B10}$$

and

$$\tan\theta = \tan\alpha\frac{1 - \frac{1}{\tan\alpha}\frac{1}{V}\frac{dV}{d\alpha}}{1 + \tan\alpha\frac{1}{V}\frac{dV}{d\alpha}}. \tag{B11}$$

Taking the derivatives of equation (B8) with respect to $\theta$, we obtain (Zhou and McMechan, 2000)

$$\frac{d\alpha}{d\theta} = \left(\frac{v}{V}\right)^2 \left[1 + \frac{1}{v}\frac{d^2 v}{d\theta^2}\right]. \tag{B12}$$

Similarly, taking the derivative of equation (B11) with respect to $\alpha$, results in

$$\frac{d\theta}{d\alpha} = \left(\frac{v}{V}\right)^2 \left[1 + 2\left(\frac{1}{V}\frac{dV}{d\alpha}\right)^2 - \frac{1}{V}\frac{d^2 V}{d\alpha^2}\right]. \tag{B13}$$

The last two equations can also be obtained by taking derivatives of equation (B5) and using equation (B1).

## APPENDIX C: COMPUTATION OF THE GROUP ANGLE

We want to use the slowness

$$p = \frac{dT}{dx} = \frac{\sin\theta}{v}, \tag{C1}$$

given in equation (56), to compute the cosine of the group angle, which is needed to compute the in-plane geometrical spreading in equation (61). Taking the derivatives of $\sin\theta = pv$ yields

$$\frac{dp}{d\theta} = \frac{\cos\theta}{v + pv'}, \tag{C2}$$

where $v' = dv/dp$. Then

$$\frac{dv}{d\theta} = \frac{dv}{dp}\frac{dp}{d\theta} = \frac{v'\cos\theta}{v + pv'}. \tag{C3}$$

From equation (B6),

$$\cos\alpha = \frac{v}{V}\left[\cos\theta - p\frac{dv}{d\theta}\right] = \frac{v^2\cos\theta}{V(v + pv')}. \tag{C4}$$

From equations (B3) and (C2) it follows that

$$\frac{v}{V} = \frac{v + pv'}{[(v + pv')^2 + \frac{v'}{v}\cos^2\theta]^{1/2}}, \tag{C5}$$

and

$$\cos\alpha = \frac{\cos\theta}{\left[1 + 2p\frac{v'}{v} + \frac{1}{v^2}\left(\frac{v'}{v}\right)\right]^{1/2}}. \tag{C6}$$

This can be expressed as

$$\cos\alpha = \left[\frac{\frac{1}{v^2} - p^2}{\frac{1}{v^2} + 2p\frac{v'}{v^3} + \left(\frac{v'}{v^3}\right)^2}\right]^{1/2}. \tag{C7}$$

We use the approximations (Stovas and Ursin, 2002)

$$\frac{1}{v^2} \simeq \frac{1}{v_0^2} - ap^2 - bp^4, \tag{C8}$$

where $a$ and $b$ are small for weak anisotropy. Then

$$\frac{v'}{v^3} = p(a + 2bp^2). \tag{C9}$$

Neglecting higher-order terms in $a$ and $b$ in equation (C8) gives

$$\begin{aligned}\cos\alpha &\simeq \left[\frac{\frac{1}{v_0^2} - p^2 - ap^2 - bp^4}{\frac{1}{v_0^2} + ap^2 + 3bp^4}\right]^{1/2} \\ &\simeq \{1 - v_0^2[(1 + 2a)p^2 + 4bp^4]\}^{1/2},\end{aligned} \tag{C10}$$

where $v_0$, $a$ and $b$ depend on the wave mode. Using the results in Appendix B of (Stovas and Ursin, 2002), $v_0$, $a$ and $b$ in the equation above are given by

$$v_0 = \alpha_0,\ a = 2\delta_M,\ b = 2\alpha_0^2\eta_M, \quad \text{for } P, \tag{C11}$$

$$v_0 = \beta_0,\ a = 2\frac{\alpha_0^2}{\beta_0^2}\eta_M,\ b = -2\alpha_0^2\eta_M, \quad \text{for } SV, \tag{C12}$$

$$v_0 = \beta_0,\ a = 2\gamma,\ b = 0, \quad \text{for } SH, \tag{C13}$$

where $\delta_M$ and $\eta_M$ are given in equations (45) and (46). The expression for $SH$-waves is exact.