

# Two theories for fluid substitution in porous rocks with aligned cracks

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## ABSTRACT

Knowledge of the correct theory for fluid substitution in anisotropic rocks is important for monitoring fluid migration in fractured reservoirs. The simplest model on which such theories can be tested is that of an isotropic, porous background rock with aligned penny-shaped cracks. In this paper the results from Thomsen's theory for fluid substitution in rocks with penny-shaped cracks are compared with the results of the combined application of linear slip theory and Brown and Korrington's fluid substitution theory. When the dry rock is modeled with linear slip theory and then "saturated" according to Brown and Korrington's equations, the anisotropic parameters of the saturated rock are numerically close to those obtained from Thomsen's theory. Thomsen's *fluid influence factor*, that determines the sensitivity of the fractures to fluid changes, is proven to be approximately inversely proportional to the *fluid storage capacity* of the fractured rock, a parameter routinely measured by well test engineers. When the fluid storage capacity of the cracks is negligible with respect to the fluid storage capacity of the isotropic background pores, Brown and Korrington's equations approach Thomsen's equations. Similar numerical results from the two theories suggest they are equivalent for most lithologies, fluid bulk moduli and reasonable crack densities. Since Brown and Korrington's theory does not require idealized crack models, the work presented in this paper can be expanded to describe the fluid substitution process in more complicated models and lower-symmetry rocks.

**Key words:** anisotropy, fluid substitution, fractures

## Introduction

Idealized models of fractured media have been frequently used in the rock physics literature to study the elasticity of naturally fractured rocks. One of the simplest models is one in which vertical, parallel fractures are embedded in an elastically isotropic background resulting in a transversely isotropic rock with a horizontal symmetry axis (HTI).

Several different effective medium theories give equivalent descriptions of the elasticity of *dry* HTI rocks (O'Connell and Budiansky, 1974; Hudson, 1981; Schoenberg and Douma, 1988). For example, assuming non-interacting fractures, Hudson's (1981) model of aligned penny-shaped cracks is equivalent to Schoenberg's linear slip model of infinite parallel planes (Bakulin et al., 2000). However, when the rock is fluid-saturated, predictions from different theories vary de-

pending on the assumptions made about the pore space connectivity, fluid viscosity and the frequency of the applied stress.

Thomsen (1995) studied how the anisotropic parameters of a porous rock with aligned penny-shaped cracks vary with changes in the bulk modulus of the saturating fluid. Thomsen proves that Hudson's (1981) theory for saturated cracks is only valid when the cracks are isolated with respect to fluid flow, as in the case of crystalline rocks with negligible primary porosity. Under those conditions a passing wave induces a large fluid pressure increase in the hydraulically disconnected cracks resulting in large changes in the anisotropic parameters with saturation. However, Thomsen argues that a correct theory for fluid substitution in fractured sedimentary rocks has to take into account the fluid pressure equilibration that takes place between hydraulically connected cracks and

non-fracture pores. The theory developed under the condition of equilibrated pore pressures was compared favorably with experimental data by Rathore et al. (1994).

The work presented here compares the theory developed by Thomsen for idealized penny-shaped crack models with a more general theory of fluid substitution developed by Brown and Korringa (1975). Brown and Korringa's (B&K) theory is a generalization of Gassmann's (1951) equations for fluid substitution in isotropic, monomineralic rocks. In contrast with Thomsen's (1995) work, B&K makes no assumptions about the strength of the anisotropy or the symmetry of the rock, nor it requires information about the shape of the fractures. Brown and Korringa's equations assume the rock is 100% fluid saturated and that when the rock is stressed, either by a static load or a passing wave, the fluid pressure is equilibrated throughout the pore space. The latter condition can be satisfied in at least three cases:

- 1) There is a single pore of arbitrary shape in the rock;
- 2) the rock has a collection of disconnected pores with the *same* shape and orientation;
- 3) all pores, with arbitrary shape and orientation, are well connected and the fluid viscosity and frequency of the applied stresses are low enough to allow equilibration of any pressure differences.

In the first case the pressure equilibration is trivial because there is only one pore in the rock. In the second case the fluid pressure increase is the same everywhere because the pore strains are identical due to the equal shape and orientation of the pores. Thomsen's theory for the fluid saturated rock is applicable to cases 2 and 3 described above. Case 2 corresponds to a rock with aligned penny-shaped cracks and no isotropic background porosity (e.g. cracked granite), and case 3 is the one most applicable to fractured sedimentary rocks.

The goal of this paper is to determine under which conditions B&K's and Thomsen's theory are identical and under which conditions they are numerically close. The analytical comparison between the theories is based on analysis of changes between the compliances of the dry and the fluid-saturated fractured rock. The numerical comparison is done by calculating the variation with fluid saturation of the anisotropic parameters  $\epsilon^{(V)}$  and  $\delta^{(V)}$  (Tsvankin, 1997) for each of the theories.

The nomenclature convention used in this work is listed at the end of the paper.

## Linear slip theory and Penny-Shaped Crack Models

The effective compliance of a fractured rock with non-interacting fractures can be obtained from two equivalent models: the *linear slip* model and *penny-shaped crack* model. In linear slip theory fractures are treated, regardless of their shape or microstructure, as planes of weakness with non-welded boundary conditions (Schoenberg and Douma, 1988; Schoenberg and Sayers, 1995). When the fractures are embedded in an otherwise isotropic background, the effective compliance matrix ( $\mathbf{S}$ ) is given by

$$\mathbf{S} = \mathbf{S}_B + \mathbf{S}_F, \quad (1)$$

where  $\mathbf{S}_B$  is the compliance matrix of the isotropic background rock and  $\mathbf{S}_F$  is the excess compliance associated with the fractures. Since the background rock is unfractured and isotropic,  $\mathbf{S}_B$  can be written in terms of the Young's modulus ( $E$ ), the shear modulus ( $\mu$ ) and the Poisson's ratio ( $\nu$ ) of the background as

$$\mathbf{S}_B = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix}. \quad (2)$$

When fractures are embedded in the isotropic background, the rock becomes more compliant to stresses applied in certain directions. These directions are determined by the fracture type and orientation, and the excess compliance the fractures contribute is described by the non-zero elements of the ( $\mathbf{S}_F$ ) matrix. If the fractures have a rotationally invariant microstructure (like penny-shaped cracks) and their normals are parallel to the  $X_1$  direction, the excess fracture compliance is given by:

$$\mathbf{S}_F = \begin{pmatrix} Z_N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_T & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_T \end{pmatrix}. \quad (3)$$

Here  $Z_N$  is the excess normal compliance that relates the fracture-normal displacements to the normal stresses applied to the fracture in the  $X_1$  direction and  $Z_T$  is the excess tangential compliance that relates tangential displacements and stresses in the  $X_2$  and  $X_3$  directions.

Notice that equation (3) from linear slip theory does not include details about the microstructure of the fractures other than the fact that they are rotationally invariant. All information about fracture shape is implicit

in the terms  $Z_N$  and  $Z_T$ . In contrast, penny-shaped crack models give explicit expressions for the excess compliance contributed by the fractures (Schoenberg and Douma, 1988; Bakulin et al., 2001).

“Penny-shaped” cracks are oblate spheroids characterized by their aspect ratios ( $\alpha_c$ ), defined as the ratio of the crack aperture ( $b$ ) divided by the crack radius ( $a$ ). The contribution of the cracks to the compliance of the rock depends on the crack density defined as  $\eta_c = Na^3/V$ , where  $N$  is the number of cracks and  $V$  is the total sample volume. In Appendix A it is shown that in a *dry* rock with vertical cracks the normal and tangential fracture compliances are, to first order in the crack density:

$$Z_N^d = \frac{16}{3} \frac{1}{E_d} (1 - \nu_d^2) \eta_c, \quad (4)$$

and

$$Z_T^d = \frac{16}{3} \frac{1}{\mu_d} \left( \frac{1 - \nu_d}{2 - \nu_d} \right) \eta_c, \quad (5)$$

where  $E_d$ ,  $\mu_d$  and  $\nu_d$  are the Young’s modulus, shear modulus and Poisson’s ratio of the dry background isotropic rock, respectively. Thomsen (1995) obtained equivalent expressions for the case of penny-shaped cracks with normals in the  $X_3$  direction [his equations (A27) and (A28)]. Adding the dry background compliance and the excess fracture compliance, the effective compliance of the dry rock ( $\mathbf{S}^d$ ) is given by:

$$\mathbf{S}^d = \begin{pmatrix} \frac{1}{E_d} + Z_N^d & -\frac{\nu_d}{E_d} & -\frac{\nu_d}{E_d} & 0 & 0 & 0 \\ -\frac{\nu_d}{E_d} & \frac{1}{E_d} & -\frac{\nu_d}{E_d} & 0 & 0 & 0 \\ -\frac{\nu_d}{E_d} & -\frac{\nu_d}{E_d} & \frac{1}{E_d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^d & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^d \end{pmatrix}. \quad (6)$$

Thomsen (1995) derived the compliance matrix of a fluid-saturated rock with penny-shaped cracks under the assumption of small crack densities and equilibrated pressures throughout the pore space. The effective compliance ( $\mathbf{S}^s$ ) of the saturated rock is given by:

$$\mathbf{S}^s = \begin{pmatrix} \frac{1}{E_s} + Z_N^s & -\frac{\nu_s}{E_s} & -\frac{\nu_s}{E_s} & 0 & 0 & 0 \\ -\frac{\nu_s}{E_s} & \frac{1}{E_s} & -\frac{\nu_s}{E_s} & 0 & 0 & 0 \\ -\frac{\nu_s}{E_s} & -\frac{\nu_s}{E_s} & \frac{1}{E_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^s & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^s \end{pmatrix},$$

where  $E_s$  and  $\nu_s$  are the Young’s modulus and Poisson’s ratio of the fluid saturated background, and  $Z_N^s$  and  $Z_T^s$  are the saturated normal and tangential fracture compliances. No distinction is made between the dry and saturated shear modulus of the isotropic background in equations (6) and (7) because it does not change with fluid saturation (Gassmann, 1951).

Thomsen’s (1995) saturated fracture compliances can be written as:

$$Z_N^s = Z_N^d \left( 1 - \frac{K_f}{K_0} \right) D_{cp}, \quad (8)$$

and

$$Z_T^s = Z_T^d, \quad (9)$$

where  $K_f$  and  $K_0$  are the bulk moduli of the saturating fluid and mineral material respectively (see Appendix A).  $D_{cp}$  is the “fluid influence factor” which accounts for the pressure equilibration between cracks and matrix pores and is defined as

$$D_{cp} = \frac{1}{1 - \frac{K_f}{K_0} + \frac{K_f}{\phi_p + \phi_c} \left( \frac{1}{K_d} - \frac{1}{K_0} + Z_N^d \right)}, \quad (10)$$

where  $\phi_c$  and  $\phi_p$  are the crack and matrix porosity, respectively, and the subscript “cp” indicates that both matrix and crack porosity are included. If the rock only has crack porosity, the fluid influence factor is obtained by setting  $\phi_p = 0$  and  $\frac{1}{K_d} = \frac{1}{K_0}$  in equation (10), which results in

$$D_c = \frac{1}{1 - \frac{K_f}{K_0} + \frac{K_f}{\phi_c} Z_N^d}, \quad (11)$$

where the subscript “c” indicates that only cracks are included.

Equations (7)-(10) are the essence of Thomsen’s (1995) contribution to understanding fluid substitution in cracked rocks under equilibrated pressure conditions. As expected for a cracked rock with HTI symmetry, the tangential fracture compliances do not change with variations of the fluid compressibility (Cardona et al., 2001). However, the normal fracture compliance ( $Z_N^s$ ) decreases with decreasing fluid compressibility.

Note that for a rock with a porous background, the fluid influence factor that determines the magnitude of the saturated fracture compliance ( $Z_N^s$ ) depends *explicitly* on the isotropic matrix porosity ( $\phi_p$ ). This results from the fact that the fluid pressure increase that stiffens the crack when a normal stress is applied depends on the total connected pore volume including the matrix porosity (Thomsen, 1995). However, in Thomsen’s formulation the saturated background compliance (e.g., component  $S_{22}^s$  in equation (7)) is *independent* of the

crack porosity even though the fluid pressure increase in the isotropic pores also depends on the total connected porosity that includes the cracks. The pressure equilibration condition throughout the total pore space suggests that *all* the compliance elements of the saturated cracked rock should depend on both the isotropic and crack porosity. This is the main distinction between Thomsen's formulation and the predictions from Brown and Korrington's theory that are presented below.

### Brown and Korrington's Theory

Brown and Korrington (1975) generalized Gassman's (1951) work by relaxing the conditions of isotropy and monomineralic rock. In their formulation, Gassman's scalar equations are replaced by equations that relate the compliance tensor of the dry and saturated rock. In this paper, only Brown and Korrington's formulation for the monomineralic case is considered.

Brown and Korrington's equations are given by:

$$S_{ijkl}^s = S_{ijkl}^d - \frac{(S_{ij\alpha\alpha}^d - S_{ij\alpha\alpha}^0)(S_{kl\alpha\alpha}^d - S_{kl\alpha\alpha}^0)}{(C_d - C_0) + (C_f - C_0)\phi_t}, \quad (12)$$

where  $S_{ijkl}^d$  and  $S_{ijkl}^s$  are the compliances of the dry and saturated rock,  $S_{ijkl}^0$  is the compliance of the mineral material,  $\phi_t$  is the total connected porosity, and  $C_f$ ,  $C_0$  and  $C_d$  are the fluid, mineral and dry rock compressibilities, respectively.

In order to be consistent with the notation used to define the fracture compliances above, B&K's equations can be written in the 6x6 matrix notation (Cardona et al., 2001). The terms with contracted indices ( $S_{ij\alpha\alpha}$ ) in the numerator of equation (12) are the sums of elements of the first three rows of the 6x6 compliance matrix. Therefore, by defining  $\psi_k$  as the sum of the first three elements of the  $k$ -th column of the compliance matrix, as shown in Figure 1a, equation (12) can be rewritten as:

$$S_{IJ}^d - S_{IJ}^s = \frac{(\psi_I^d - \psi_I^0)(\psi_J^d - \psi_J^0)}{(C_d - C_0) + (C_f - C_0)\phi_t}. \quad (13)$$

Better understanding of the fluid substitution process can be achieved by rewriting equations (13) as

$$S_{IJ}^s = S_{IJ}^d - (\psi_I^d - \psi_I^0) \frac{\partial p_f}{\partial \sigma}, \quad (14)$$

where

$$\frac{\partial p_f}{\partial \sigma} = \frac{\psi_I^d - \psi_I^0}{(C_d - C_0) + (C_f - C_0)\phi_t}. \quad (15)$$

Here  $\frac{\partial p_f}{\partial \sigma}$  is the gradient of the fluid pressure ( $p_f$ ) induced by the variation of the confining stress ( $\Delta\sigma$ ) (Brown and Korrington, 1975).

If the rock pores are empty ( $C_f \rightarrow \infty$ ), there will

$$S_{ij} = \begin{array}{c} \begin{array}{cccccc} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ \hline \begin{array}{|c|c|c|} \hline S_{11} & S_{12} & S_{13} \\ \hline S_{12} & S_{22} & S_{23} \\ \hline S_{13} & S_{23} & S_{33} \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline S_{14} & S_{15} & S_{16} \\ \hline S_{24} & S_{25} & S_{26} \\ \hline S_{34} & S_{35} & S_{36} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|} \hline S_{14} & S_{24} & S_{34} \\ \hline S_{15} & S_{25} & S_{35} \\ \hline S_{16} & S_{26} & S_{36} \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline S_{44} & S_{45} & S_{46} \\ \hline S_{45} & S_{55} & S_{56} \\ \hline S_{46} & S_{56} & S_{66} \\ \hline \end{array} \end{array} \end{array}$$

**Figure 1.** Compliance matrix of general symmetry indicating elements that contribute to the sums  $\psi_k$  in equation (13).

be no variation in the fluid pressure when a differential confining stress ( $\Delta\sigma$ ) is applied. Hence, the term  $\frac{\partial p_f}{\partial \sigma}$  in equation (15) is zero and the saturated compliance ( $S_{ijkl}^s$ ) is equal to the dry compliance ( $S_{ijkl}^d$ ). When the rock is fluid-saturated, the confining stress  $\Delta\sigma$  strains the pore volume, inducing a fluid pressure change equal to  $\frac{\partial p_f}{\partial \sigma} \Delta\sigma$  that reduces the compliance of the rock [see equation (14)].

### Total storage capacity and the fluid influence factor

The change in fluid pressure depends on the denominator of equation (15), which is equal to the total storage capacity of the pore system. The total storage capacity relates the variations in pore fluid volume to variations in fluid pressure and is defined as the product of the total porosity ( $\phi_t$ ) and the total pore system compressibility ( $C_t$ ) (see Appendix B). The induced pressure is inversely proportional to  $\phi_t C_t$  because a rock with a large storage capacity can accommodate the fluid displaced by the pore volume change with only a small pressure variation.

In the case of the rock with isotropic background porosity ( $\phi_p$ ) and crack porosity ( $\phi_c$ ), the total storage capacity is the sum of the storage capacities of the isotropic pores and crack pores:

$$\phi_t C_t = \phi_p C_p + \phi_c C_c. \quad (16)$$

Using the compliance matrix of the dry fractured rock defined in equations (4)-(6), it can be shown that the storage capacity of the isotropic porosity and penny-shaped cracks are given by:

$$\phi_p C_p = \left( \frac{1}{K_d} - \frac{1}{K_0} \right) + \left( \frac{1}{K_f} - \frac{1}{K_0} \right) \phi_p, \quad (17)$$

and

$$\phi_c C_c = Z_N^d + \left( \frac{1}{K_f} - \frac{1}{K_0} \right) \phi_c, \quad (18)$$

respectively (Appendix B).

From equations (16)-(18) it is straightforward to prove that the inverse of the total storage capacity is:

$$\frac{1}{\phi_t C_t} = \frac{\frac{K_f}{\phi_t}}{1 - \frac{K_f}{K_0} + \frac{K_f}{\phi_p + \phi_c} \left( \frac{1}{K_d} - \frac{1}{K_0} + \frac{K_d}{K_0} Z_N^d \right)}. \quad (19)$$

Comparison of the previous expression to the fluid influence factor [equation (16)] indicates that the inverse of the total storage capacity is approximately proportional to Thomsen's fluid influence factor, i.e.:

$$\frac{1}{\phi_t C_t} \approx D_{cp} \frac{K_f}{\phi_t}. \quad (20)$$

The approximate proportionality results from the extra  $\frac{K_d}{K_0}$  term multiplied with the dry normal compliance ( $Z_N^d$ ) in equation (19) that is not present in the expression of the fluid influence factor. For rocks with only crack porosity ( $\phi_p = 0$ ),  $\frac{K_d}{K_0} = 1$  and the approximate expression (21) becomes exact:

$$\frac{1}{\phi_c C_c} = D_c \frac{K_f}{\phi_c}. \quad (21)$$

Equations (20) and (21) indicate that Thomsen's fluid influence factor can be interpreted as the inverse of the normalized pore system compressibility ( $\frac{1}{K_f C_t}$ ) or as being inversely proportional to the storage capacity of the pore system.

### Saturated compliance for non-porous background

For the case of a non-porous background matrix, substitution of the dry compliance (6) into B&K's equations indicates that only the  $S_{11}^d$  component of the dry rock compliance varies with saturation (see Appendix C). Since the background matrix is non-porous, the change with saturation is due exclusively to the change in the fracture normal compliance.  $S_{11}^s$  is given by:

$$S_{11}^s = \frac{1}{E_0} + Z_N^s, \quad (22)$$

where  $E_0$  is the mineral Young's modulus, and

$$Z_N^s = \frac{Z_N^d \left( 1 - \frac{K_f}{K_0} \right)}{1 - \frac{K_f}{K_0} + \frac{K_f}{\phi_c} Z_N^d} \equiv Z_N^d \left( 1 - \frac{K_f}{K_0} \right) D_c. \quad (23)$$

Comparing the previous expression to equation (8) shows that Thomsen's (1995) saturated normal fracture compliance is identical to the one obtained from B&K. Hence, both theories are identical for the case of a cracked rock with no isotropic background porosity.

However, this is not the case for a rock with a porous background matrix, as shown below.

### Saturated compliance for porous background

In a rock with isotropic background porosity the induced fluid pressure variation will depend on the storage capacity of both the isotropic and crack pores. Therefore, by virtue of equations (14) and (15), *all* the elements of the compliance matrix that change with saturation will depend *explicitly* on the crack porosity. The expressions for the compliance elements that change with saturation are given by:

$$S_{11}^s = \underbrace{\frac{1}{E_d} - \frac{1}{9} \left( \frac{\phi_p}{K\phi_p} \right)^2}_{\approx \frac{1}{E_s}} + \frac{\left( 1 - \frac{K_f}{K_0} + \frac{K_f}{3K\phi_p} \frac{\phi_p}{\phi_t} \right) Z_N^d}{\frac{K_f}{\phi_t} (\phi_t C_t)}, \quad (24)$$

$$S_{12}^s = S_{13}^s = - \underbrace{\frac{\nu_d}{E_d} - \frac{1}{9} \left( \frac{\phi_p}{K\phi_p} \right)^2}_{\approx -\frac{\nu_s}{E_s}} - \frac{\frac{K_f}{3K\phi_p} \frac{\phi_p}{\phi_t} Z_N^d}{\frac{K_f}{\phi_t} (\phi_t C_t)}, \quad (25)$$

$$S_{22}^s = S_{33}^s = \frac{1}{E_d} - \frac{1}{9} \left( \frac{\phi_p}{K\phi_p} \right)^2 \approx \frac{1}{E_s}, \quad (26)$$

$$S_{23}^s = - \frac{\nu_d}{E_d} - \frac{1}{9} \left( \frac{\phi_p}{K\phi_p} \right)^2 \approx -\frac{\nu_s}{E_s}, \quad (27)$$

where  $\frac{1}{K\phi_p}$  is the compaction compressibility of the isotropic background pores (see Appendix C).

Equations (24)-(27) indicate that in general it is not possible to express the saturated compliance of a cracked rock with porous background exactly as the sum of the saturated isotropic background and a saturated fracture compliance (other than the case  $\phi_p = 0$  discussed above). However, the symbols " $\approx$ " in equations (24)-(27) mark the terms that approach the background's saturated values when the storage capacity of the cracks ( $\phi_c C_c$ ) is negligible with respect to the storage capacity of isotropic pores ( $\phi_p C_p$ ) (see Appendix C). In this approximation the saturated compliance is given by:

$$S^s \approx \begin{pmatrix} S_{11}^s & S_{12}^s & S_{12}^s & 0 & 0 & 0 \\ S_{12}^s & \frac{1}{E_s} & -\frac{\nu_s}{E_s} & 0 & 0 & 0 \\ S_{12}^s & -\frac{\nu_s}{E_s} & \frac{1}{E_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^s & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} + Z_T^s \end{pmatrix}.$$

(28)

where  $Z_T^s = Z_T^d$  (no approximation). In the expression above  $S_{11}^s$  and  $S_{12}^s$  are:

$$S_{11}^s \approx \frac{1}{E_s} + Z_N^d \left( 1 - \frac{K_f}{K_0} + \frac{K_f}{3K_{\phi_p}} \frac{\phi_p}{\phi_t} \right) D_p, \quad (29)$$

and

$$S_{12}^s \approx -\frac{\nu_s}{E_s} - \frac{\frac{\phi_p}{3K_{\phi_p}} Z_N^d}{\phi_t C_t}, \quad (30)$$

where  $D_p$  is the fluid influence factor without including the storage capacity contribution of the cracks.

Equations (28)-(30) indicate that in the case of negligible crack storage capacity the saturated compliance resulting from B&K's equations is similar in form to the one obtained by Thomsen (1995) [equations (7)-(10)]. The main difference is in the elements  $S_{12}^s$  and  $S_{13}^s$  which have an extra term proportional to  $Z_N^d$  [equation (30)].

Although the arguments above prove that there are differences between the two theories, the important practical question is whether these differences result in measurable discrepancies in the prediction of the anisotropic parameters of the fractured rock. If in situ rocks are weakly anisotropic ( $Z_N^d$  is small compared to  $\frac{1}{E_d}$ ) and have small fracture (crack) storage capacities, we can expect Thomsen's (1995) approach to be numerically equivalent to the B&K theory. The numerical results discussed in the next section illustrate the differences between theories.

### Numerical results

Wave propagation in a HTI medium, such as a rock with aligned vertical cracks, is most conveniently described by the vertical P- and S-wave velocities and three anisotropic coefficients defined as:

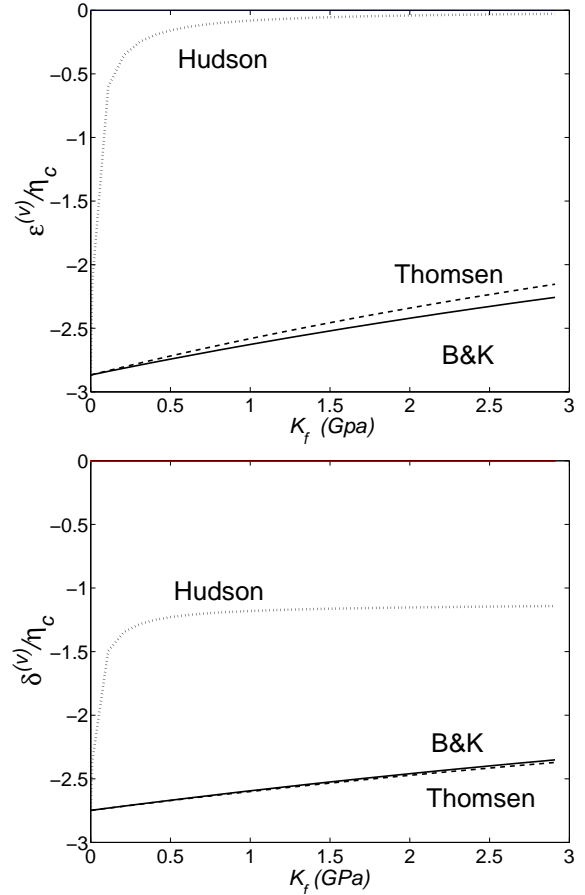
$$\epsilon^{(V)} \equiv \frac{c_{11} - c_{33}}{2c_{33}}, \quad (31)$$

$$\delta^{(V)} \equiv \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}, \quad (32)$$

$$\gamma^{(V)} \equiv \frac{c_{66} - c_{44}}{2c_{44}}, \quad (33)$$

where the  $c_{ij}$  correspond to the elements of the 6x6 stiffness matrix ( $\mathbf{c}$ ) (Tsvankin, 1997).

The numerical comparison between Thomsen's (1995) and Brown and Korringa's fluid substitution theories is done by calculating the  $\epsilon^{(V)}$  and  $\delta^{(V)}$  parameters as a function of the bulk modulus of the saturating fluid ( $K_f$ ). Since under conditions of equilibrated pore pressures the  $\gamma^{(V)}$  coefficient does not change with saturation (Thomsen, 1995), it is not included in this comparison.



**Figure 2.**  $\epsilon^{(V)}$  and  $\delta^{(V)}$  as a function of  $K_f$  for a calcite matrix with  $\phi_p = 0.05$ ,  $\phi_c = 10^{-4}$ ,  $\eta_c = 0.05$  and  $\alpha_c = 4 \times 10^{-4}$ .

The procedure to obtain  $\epsilon^{(V)}$  vs.  $K_f$  and  $\delta^{(V)}$  vs.  $K_f$  curves is as follows:

- 1) Model the **dry** isotropic background compliance ( $\mathbf{S}_B$ ) for a given mineral and an isotropic porosity ( $\phi_p$ ).
- 2) Add the excess fracture compliances of the **dry** penny-shaped cracks ( $\mathbf{S}_F$ ).
- 3) Apply a fluid substitution theory to obtain the saturated rock compliance ( $\mathbf{S}^s$ ).
- 4) Invert the saturated compliance matrix to obtain the saturated stiffness matrix ( $\mathbf{c}^s$ ).
- 5) Calculate  $\epsilon^{(V)}$  and  $\delta^{(V)}$  from the stiffness of the fluid-saturated rock ( $\mathbf{c}^s$ ).

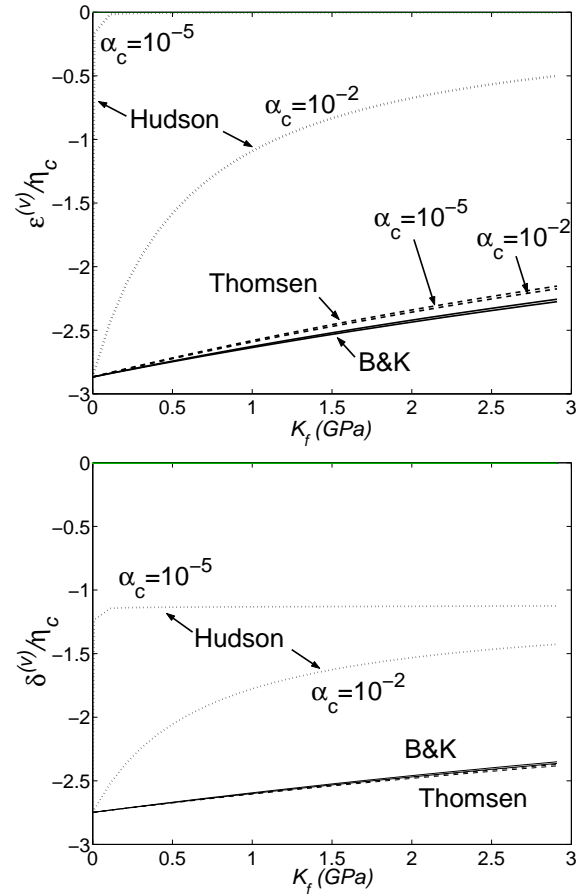
Figures 2-4 show  $\epsilon^{(V)}$  and  $\delta^{(V)}$  as a function of  $K_f$  for several combinations of isotropic porosity ( $\phi_p$ ), crack porosity ( $\phi_c$ ) and crack aspect ratio ( $\alpha_c$ ). Numerical analysis indicates that the variation of  $\epsilon^{(V)}$  and  $\delta^{(V)}$  with  $K_f$  depends weakly on the elastic properties of the mineral materials that compose typical sedimentary rocks. Feldspars, limestones and dolomites with Poisson's ratio varying between 0.25 and 0.35 have almost identical  $\epsilon^{(V)}$  vs.  $K_f$  and  $\delta^{(V)}$  vs.  $K_f$  dependence. Hence, all results shown in the figures below assume a fixed isotropic limestone matrix with bulk and shear moduli of  $K_0 = 77 \text{ GPa}$  and  $\mu_0 = 32 \text{ GPa}$ , respectively.

Figure 2 shows the change with saturation of the  $\epsilon^{(V)}$  and  $\delta^{(V)}$  parameters normalized by the crack density. The results are for the case of a cracked limestone rock with an isotropic porosity of  $\phi_p = 0.05$ , a crack porosity of  $\phi_c = 10^{-4}$ , and a crack density and aspect ratio of  $\eta_c = 0.05$  and  $\alpha_c = 4 \times 10^{-4}$ , respectively. Hudson's (1981) predictions are shown for comparison.

As expected from the condition of equilibrated pore pressure, both Thomsen's and B&K theory predict smaller changes with saturation in the anisotropies than Hudson's (1981) theory. The difference between Thomsen's and B&K's results is small for both parameters and almost null for the  $\delta^{(V)}$  coefficient.

Thomsen (1995) noted that under conditions of equilibrated pore pressures the fluid variation of the anisotropies is not sensitive to changes in the aspect ratio of the cracks. Figure 3 indicates that Thomsen's observation is consistent with B&K's results, where there is almost no variation in the  $\epsilon^{(V)}$  vs.  $K_f$  or  $\delta^{(V)}$  vs.  $K_f$  curves when the aspect ratio varies three orders of magnitude.

As predicted by equations (24)-(27), Figure 4 indicates that the small numerical differences between the predictions from Thomsen's and Brown and Korrington's theory disappear as the isotropic porosity of the background rock ( $\phi_p$ ) becomes smaller. The fact that in both theories the change in  $\epsilon^{(V)}$  and  $\delta^{(V)}$  with saturation is larger when  $\phi_p$  decreases may seem counter-intuitive at first. However, this can be understood recalling the role of the storage capacity in the fluid substitution process. If the rock under stress has a large isotropic porosity ( $\phi_p$ ), the rock has a large storage capacity to accommodate the fluid displaced from the cracks with only a small fluid pressure increase. If the porosity is small, the reduced storage capacity results in a larger fluid pressure increase that stiffens all pores (including the cracks) and reduces the anisotropies  $\epsilon^{(V)}$  and  $\delta^{(V)}$ .

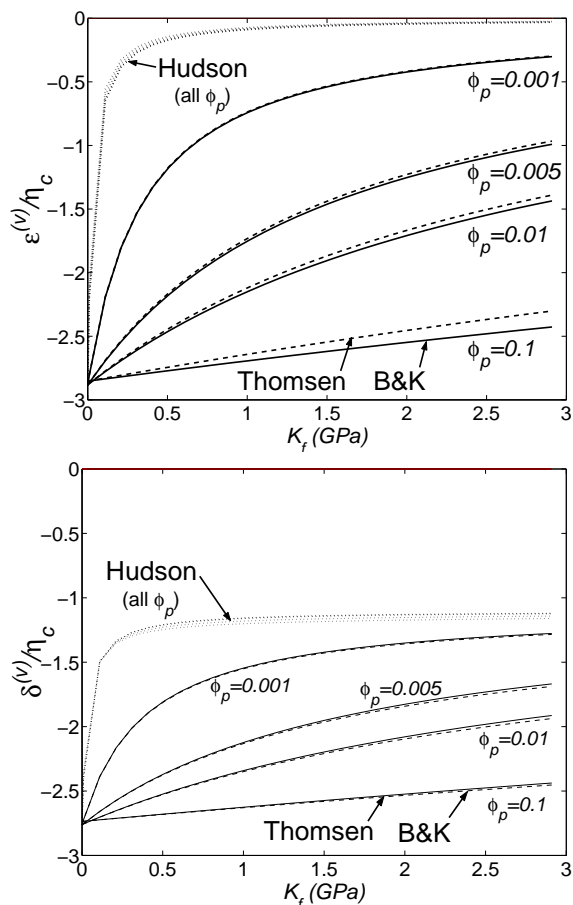


**Figure 3.** Sensitivity of  $\epsilon^{(V)}$  and  $\delta^{(V)}$  to the crack aspect ratio for a calcite matrix with  $\phi_p = 0.05$  and  $\eta_c = 0.05$ .  $\alpha_c = 10^{-5}$  corresponds to  $\phi_c = 2.1 \times 10^{-6}$  and  $\alpha_c = 10^{-2}$  corresponds to  $\phi_c = 2.1 \times 10^{-3}$

## Discussion and Conclusions

This work proves that Thomsen's (1995) theory for fluid saturated, cracked rocks under equilibrated pore pressure conditions is consistent with Brown and Korrington's (1975) theory of fluid substitution. When the rock has *only* crack porosity, the predictions from both theories are analytically *identical*. When the rock has both background isotropic porosity and crack porosity, the predictions differ analytically but numerically they are almost identical.

This work shows that the fluid influence factor defined by Thomsen (1995) can be re-interpreted as being inversely proportional to the total storage capacity of the rock. Since the total storage capacity is a parameter estimated regularly by well test engineers from pressure-transient experiments (Cardona et al., 2002), it can provide information on how sensitive the normal fracture



**Figure 4.**  $\epsilon^{(V)}$  and  $\delta^{(V)}$  as a function of  $K_f$  for several values of the isotropic porosity. Curves are calculated for a calcite matrix with  $\eta_c = 0.05$  and  $\alpha_c = 4 \times 10^{-4}$ .

compliance is to changes in fluid saturation. When the isotropic porosity is small, the reduced storage capacity results in a large fluid influence factor and large changes in the normal fracture compliance with saturation.

For a rock with background porosity, the discrepancies between Thomsen (1995) and B&K are significant only if the normal fracture compliance is large with respect to the background compliance and the fracture storage capacity is large compared to the background storage capacity [equations (24)-(27)]. However, reservoir engineers usually acknowledge that in double-porosity reservoirs (with both fracture and non-fracture porosity) the fracture storage capacity is negligible compared to the background storage capacity. If naturally fractured rocks are also assumed to be weakly anisotropic, we may expect that Thomsen's and B&K's theories are roughly equivalent in most practical cases.

Even though Brown and Korrington's theory is almost equivalent to Thomsen's (1995) formulation, it has the

advantage of being more general. Brown and Korrington's equations are valid for arbitrary symmetry and arbitrary strength of anisotropy. No assumptions about the detailed structure of the fractures are required and it could apply to rocks with an arbitrary number of fracture sets as long as the fluid pressure is equilibrated throughout the pore space. Therefore, a more general approach to modeling the elasticity of fractured rocks is to model the dry rock with any appropriate theory (e.g. the linear slip theory) and then calculate the fluid saturated values from Brown and Korrington's equations.

The linear slip theory is convenient for modeling dry rocks because it does not require detailed information about the microstructure of the fractures and is valid for several non-interacting fracture sets (Bakulin et al., 2000). The condition of non-interacting fractures can be met assuming the fracture volume density is small. When the rock is fluid-saturated, all interactions between pores due to the condition of equilibrated pore pressures is taken into account through Brown and Korrington's equations.

For naturally fractured rocks it may be argued that the non-interacting fracture condition is in conflict with the condition of equilibrated pore pressures since pressure equilibrium requires hydraulic connectivity between all pores. This argument can be countered assuming that the non-interacting fractures are connected through the background pores or assuming that every *real* dry rock has a linear slip equivalent that can be used to model it. Ultimately it will be the application of the theory to real data that will determine its validity.

## Acknowledgments

I am greatly indebted to Drs. Vladimir Grechka, Ilya Tsvankin, Andrey Bakulin and Michael Batzle for their helpful comments and suggestions. I am also grateful to the members of the A(nisotropy)-Team of the CWP for helpful discussions. This work has been supported by the Reservoir Characterization Project and the Center for Wave Phenomena at the Colorado School of Mines.

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**Table 1.** Nomenclature

$C$	compressibility
$c_{ij}$	stiffness matrix
$D$	fluid influence factor
$E$	Young's Modulus
$K$	bulk modulus
$p$	pressure
$S_{ij}$	compliance matrix
$S_{ijkl}$	compliance matrix
$S_B$	background compliance matrix
$S_F$	excess fracture compliance matrix
$V_p$	pore volume
$Z_N$	normal fracture system compliance
$Z_T$	tangential fracture system compliance
$\alpha_c$	crack aspect ratio
$\eta_c$	crack density
$\nu$	Poisson's ratio
$\mu$	shear modulus
$\phi$	porosity
$\rho$	fluid density
$\sigma$	confining stress
<i>Subscripts</i>	
$d$	dry
$s$	saturated
$0$	mineral
$c$	crack
$f$	fluid
$t$	total

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## APPENDIX A: Fluid saturation in Thomsen's model of cracked porous rock

Appendix A of Thomsen's (1995) paper gives expressions for the elements of the compliance matrix of the fluid-saturated cracked rock for the case of aligned cracks with normals in the  $X_3$  direction. If the cracks are vertical with normals in the  $X_1$  direction, the crack-normal compliance element corresponds to  $S_{11}^s$  and the crack-tangential compliance element corresponds to  $S_{55}^s = S_{66}^s$ . Therefore,  $S_{11}^s$ ,  $S_{55}^s$  and  $S_{66}^s$  are given by

$$S_{11}^s = \frac{1}{E_s} + \left(1 - \frac{K_f}{K_0}\right) \frac{A_c \eta_c}{K_d} D_{cp}, \quad (\text{A1})$$

and

$$S_{55}^s = S_{66}^s = \frac{1}{\mu} + \frac{B_c \eta_c}{\mu}, \quad (\text{A2})$$

where  $K_f$ ,  $K_d$  and  $K_0$  are the the bulk moduli of the fluid, dry isotropic background rock and mineral, respectively.  $E_s$  and  $\mu$  are the Young's modulus and shear modulus of the saturated isotropic background rock, respectively, and  $\eta_c$  is the crack density. The constants  $A_c$  and  $B_c$  are

$$A_c = \frac{16}{9} \left( \frac{1 - \nu_d^2}{1 - 2\nu_d} \right), \quad (\text{A3})$$

and

$$B_c = \frac{16}{3} \left( \frac{1 - \nu_d}{2 - \nu_d} \right), \quad (\text{A4})$$

where  $\nu_d$  is the Poisson's ratio of the dry isotropic background rock.

Since the second term in equations (A1) and (A2) are crack-dependent terms added to the background compliances  $\frac{1}{E_s}$  and  $\frac{1}{\mu}$ , by definition they are the normal and tangential excess fracture compliances. Therefore, expressing  $K_d$  as a function of  $E_d$  and  $\nu_d$ , and substituting it in equation (A1) we obtain:

$$Z_N^s = Z_N^d \left(1 - \frac{K_f}{K_0}\right) D_{cp}, \quad (\text{A5})$$

and

$$Z_T^s = \frac{16}{3} \frac{1}{\mu_d} \left( \frac{1 - \nu_d}{2 - \nu_d} \right) \eta_c = Z_T^d, \quad (\text{A6})$$

where  $Z_N^d$  is the fracture compliance when  $K_f = 0$  defined as:

$$Z_N^d = \frac{16}{3} \frac{1}{E_d} (1 - \nu_d^2) \eta_c. \quad (\text{A7})$$

## APPENDIX B: The role of storage capacity in fluid substitution

The storage capacity can be defined from the equation of continuity for fluid flow in a porous medium. Due to mass conservation, the change in mass per unit time between

the in-going flow and out-going flow has to be equal to the amount of fluid stored per unit time in the pores. This condition can be written as:

$$\nabla(\rho\vec{v}) = \frac{\partial(\rho\phi_t)}{\partial t}, \quad (\text{B1})$$

where  $\rho$  is the fluid density,  $\vec{v}$  is the fluid flow velocity, and  $\phi_t$  is the total porosity. Because fluid pressure variations are easier to monitor than fluid mass variations, the right side of equation (B1) can be written as

$$\frac{\partial(\rho\phi_t)}{\partial t} = \rho \left[ \frac{1}{\rho} \frac{\partial(\rho\phi_t)}{\partial p_f} \right] \frac{\partial p_f}{\partial t}, \quad (\text{B2})$$

where the term in the brackets is defined as the storage capacity of the rock. Evaluating the term in the brackets, it can be shown that

$$\frac{1}{\rho} \frac{\partial(\rho\phi_t)}{\partial p_f} = \phi_t \underbrace{(C_f + C_{pp})}_{C_t}, \quad (\text{B3})$$

where  $C_f$  is the fluid compressibility,  $C_t$  is the total pore system compressibility and  $C_{pp}$  is the pore space compressibility at a constant confining pressure defined as:

$$C_{pp} = \frac{1}{V_p} \left( \frac{\partial V_p}{\partial p_f} \right). \quad (\text{B4})$$

Zimmerman (1991) proved that

$$c_{pp} = \left( \frac{1}{\hat{K}_d} - \frac{1 + \phi_t}{K_0} \right) \frac{1}{\phi_t}, \quad (\text{B5})$$

where  $\frac{1}{\hat{K}_d}$  is the bulk compressibility of the dry rock (this applies to any monomineralic rock in which the mineral material is isotropic). In the case of the HTI symmetry rock it can be proven, using the dry compliance matrix (6) and the definition of the bulk compressibility ( $\frac{1}{\hat{K}_d} = S_{\alpha\alpha\beta\beta}^d$ ), that the bulk compressibility of the fractured rock is given by:

$$\frac{1}{\hat{K}_d} = Z_N^d + \frac{1}{K_d}, \quad (\text{B6})$$

where  $\frac{1}{K_d}$  is the compressibility of the dry isotropic background. Substitution of equation (B5) into equation (B3) proves that the total storage capacity of the rock can be written as:

$$\phi_t C_t = \left( \frac{1}{\hat{K}_d} - \frac{1}{K_0} \right) + \left( \frac{1}{K_f} - \frac{1}{K_0} \right) \phi_t, \quad (\text{B7})$$

which is equal to the denominator on the right side of Brown and Korrington's equation (13).

Substituting equation (B6) into equation (B7) results in:

$$\phi_t C_t = \underbrace{\left( \frac{1}{\hat{K}_d} - \frac{1}{K_0} \right)}_{\equiv \phi_p C_p} + \left( \frac{1}{K_f} - \frac{1}{K_0} \right) \phi_p +$$

$$\underbrace{Z_N^d + \left( \frac{1}{K_f} - \frac{1}{K_0} \right) \phi_c}_{\equiv \phi_c C_c}, \quad (\text{B8})$$

where the first term is the storage capacity of the isotropic background pores ( $\phi_p C_p$ ) and the second term is the storage capacity of the crack pores ( $\phi_c C_c$ ).

### APPENDIX C: Change of the compliance matrix with saturation in Brown and Korrington's theory

The change with saturation of the compliance matrix can be calculated from the B&K equation in two-index notation:

$$S_{IJ}^d - S_{IJ}^s = \frac{(\psi_I^d - \psi_I^0)(\psi_J^d - \psi_J^0)}{(\hat{C}_d - C_0) + (C_f - C_0)\phi_t}. \quad (\text{C1})$$

Using the dry compliance matrix (6) of the fractured rock and the compliance matrix of the isotropic mineral material, the terms in the numerator of the right side of equation (C1) are given by

$$(\psi_1^d - \psi_1^0) = Z_N^d + \frac{1}{3} \left( \frac{1}{K_d} - \frac{1}{K_0} \right), \quad (\text{C2})$$

$$(\psi_2^d - \psi_2^0) = (\psi_3^d - \psi_3^0) = \frac{1}{3} \left( \frac{1}{K_d} - \frac{1}{K_0} \right), \quad (\text{C3})$$

and

$$(\psi_4^d - \psi_4^0) = (\psi_5^d - \psi_5^0) = (\psi_6^d - \psi_6^0) = 0, \quad (\text{C4})$$

where the elements  $\psi_I$ 's are defined in the main text. From equations (C2)-(C4) it is clear that the only compliance elements that will change with saturation are  $S_{11}^s$ ,  $S_{22}^s = S_{33}^s$ ,  $S_{12}^s = S_{13}^s$ , and  $S_{23}^s$ .

From equations (C2)-(C4) and noting that the denominator on the right side of equation (C1) is the total storage capacity of the rock (Appendix B), the saturated compliances can be written as

$$S_{11}^s = \frac{1}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_t C_t} + \frac{\left( 1 - \frac{K_f}{K_0} + \frac{K_f}{3K_{\phi_p}} \frac{\phi_p}{\phi_t} \right) Z_N^d}{\frac{K_f}{\phi_t} (\phi_t C_t)}, \quad (\text{C5})$$

$$S_{22}^s = S_{33}^s = \frac{1}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_t C_t}, \quad (\text{C6})$$

$$S_{12}^s = S_{13}^s = - \frac{\nu_d}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_t C_t} - \frac{\frac{K_f}{3K_{\phi_p}} \frac{\phi_p}{\phi_t} Z_N^d}{\frac{K_f}{\phi_t} (\phi_t C_t)}, \quad (\text{C7})$$

$$S_{23}^s = - \frac{\nu_d}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_t C_t}. \quad (\text{C8})$$

Here  $\frac{1}{K_{\phi_p}}$  is the compaction compressibility of the isotropic background pores that relates the dry isotropic compressibility and the mineral compressibility according to

$$\frac{1}{K_{\phi_p}} = \frac{1}{\phi_p} \left( \frac{1}{K_d} - \frac{1}{K_0} \right). \quad (C9)$$

From equations (C5)-(C8) it can be seen that if the rock has no isotropic porosity ( $\phi_p = 0$ ), then  $S_{22}^s, S_{33}^s, S_{12}^s, S_{13}^s, S_{33}^s$  will take the value of the isotropic background mineral and only  $S_{11}^s$  will change with saturation. In this case B&K's equations are *identical* to Thomsen's (1995) formulation, as explained in the main text.

When the storage capacity of the cracks is negligible, the total storage capacity is approximately equal to the storage capacity of the isotropic background pores, i.e.  $\phi_t C_t \approx \phi_p C_p$ . Under this approximation, the terms marked underneath in equations (C5) and (C6) are approximately equal to  $\frac{1}{E_s}$ , and the terms in equations (C7) and (C8) are approximately equal to  $-\frac{\nu_s}{E_s}$ . This results from the application of B&K's theory to a dry isotropic rock in which  $\frac{1}{E_d}$  and  $-\frac{\nu_d}{E_d}$  vary with saturation as

$$\frac{1}{E_s} = \frac{1}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_p C_p}, \quad (C10)$$

and

$$-\frac{\nu_s}{E_s} = -\frac{\nu_d}{E_d} - \frac{\frac{1}{9} \left( \frac{\phi_p}{K_{\phi_p}} \right)^2}{\phi_p C_p}. \quad (C11)$$

Therefore, in the approximation of negligible crack storage capacity, the compliance of the saturated fractured rock can be written as the sum of a background isotropic compliance and an excess fracture compliance. However, due to the extra fracture term in the  $S_{12}^s$  and  $S_{13}^s$  elements, the saturated compliance predicted by B&K's theory is not *analytically* identical to Thomsen's (1995) formulation.

