The breakdown of wave diffusion in 2D due to loops

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ABSTRACT

Diffusion of multiply scattered waves is a phenomenological model that can fail for a variety of reasons. Most studies that apply the diffusion approximation use the solution for an infinite uniform halfspace (Aki and Chouet, 1975; Wegler and Luhr, 2001). Though this overly simplified model yields fruitful results, Margerin *et al* (1998) have shown that a greater understanding of both intrinsic and scattering attenuation can be obtained by limiting the scattering to a layer (e.g., the crust of the Earth). Here we explore the validity of the diffusion approximation for a finite-sized scattering medium. Since the diffusion approximation includes only self-avoiding scattering paths, it underestimates the intensity emerging from a finite-sized scattering medium at late times. We show this by performing a numerical experiment and comparing the results to an analytic expression that includes only self-avoiding paths. We develop a theory to quantify this underestimate based on counting possible scattering paths between point scatterers. Interference phenomena, due to closed loops, are incorporated into the theory in a way similar to the calculation of coherent backscattering, or weak localization. The behavior gives insight into the late-time dynamics of multiple scattering.

Key words: diffusion approximation, self-avoiding random walk, multiple scattering

Introduction

It has been said that you see what you want to see and hear what you want to hear. Nowhere is this more true than within the field of seismology. Everyday, odd or exceptional data are muted or filtered out so that the remaining data are amenable to the chosen method of processing. For example, since the scattering coefficient for elastic waves is dependent on the square of frequency, applying a lowpass filter to the data ensures that the filtered data are in the single scattering regime. As a result, migration algorithms can be applied to the filtered data in a meaningful way.

An example of the dramatic action of frequency filtering is shown in Fig. 1. An earthquake in Greece produces the vertical ground motion, shown in the top panel, at a station in The Netherlands. After highpass filtering the seismic trace, the signal is a rapidly decaying, seemingly random burst of energy (middle panel). Only after accentuating the low frequency content do the P-, S-, and Rayleigh arrivals show up (bottom panel). These waves can be adequately modeled by considering a radially symmetric Earth and are therefore said to be deterministic. Furthermore, with enough independently recorded deterministic arrivals, an imaging procedure can be posed that finds the smoothed earth model that best fits the filtered data. A good initial guess for such a process would probably be a radially symmetric Earth.

A comprehensive understanding of the subsurface demands not only an explanation of the well-defined aspects, but also the complicated high-frequency parts of the signal. Unfortunately, the information is generally too scant to indicate exactly from where in the subsurface the complex high frequency signal comes. As a result, a statistical treatment of the data can be helpful. By statistical, we mean that average quantities can be understood, such as the average total intensity of the wavefield; however "there is a very important and fundamental truth about random systems we must always

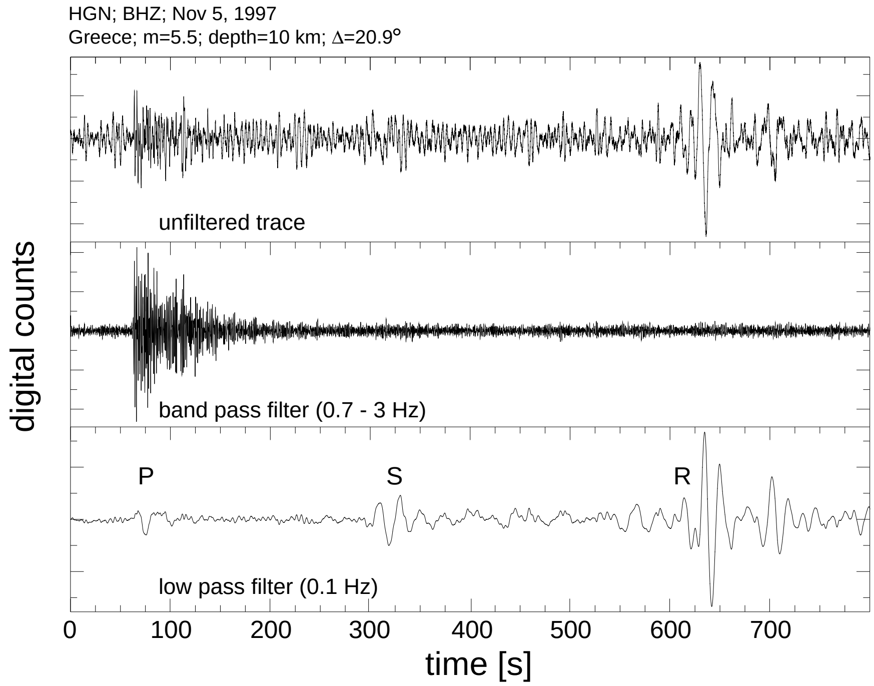

Figure 1. The vertical ground motion, recorded in the Netherlands from an earthquake in Greece, is displayed in the top panel. Also shown are the band-passed filtered seismogram containing frequencies from 0.7 Hz to 3 Hz (middle panel) and the low-passed filtered seismogram with frequencies less than 0.1 Hz (bottom panel). (Snieder, 1999)

keep in mind: no real atom is an average atom, nor is an experiment done on an ensemble of samples" (Anderson, 1978).

For sufficient multiple scattering, the average total intensity can be shown to obey a diffusion equation (Tourin *et al*, 2000) and the waves are said to be *diffuse waves.* Diffuse seismic waves have been observed in the study of volcanoes by both passive and active means (Aki and Ferrazzini, 2000; Wegler and Luhr, 2001). The application of diffuse-wave concepts was desirable in these cases because of the following complications:

(i) the geology was not predominately a 1D layering,
(ii) there was no clear onset of P- or S-waves,
(iii) there was little coherence between receivers.

Strong scattering observed at these volcanoes, however, may have put the standard picture of diffusion in jeopardy, as noted by both authors. One purpose of this paper is to explore the limitations of this theory.

Statistical methods based on the diffusion of elastic waves should be ideal for late-time recordings in seismic exploration. One advantage is that a multiple scattering model can allow separation of scattering and intrinsic attenuation (Wu, 1985). In an ultrasonic experiment, Scales and Van Wijk (2001) have shown that the intensity transmitted through a 1D randomly layered medium obeys a diffusion equation with an attenuation term.

Recent advances within the fields of acoustics, optics, and medical imaging have relied on diffuse-wave concepts (Boas *et al*, 1995; Yodh and Chance, 1995; Virmont and Ledanois, 1998). Pertinent material properties can be extracted from waves that have scattered many times. For example, Schirrer *et al* (1997) investigated the formation of cracks in glassy polymers by exploiting the interference between counter-propagating light waves.

Recurrent Multiple Scattering

Interference has played a vital role in an upwelling of interest in the multiple scattering of classical waves. Its consideration led to the discovery of coherent backscattering (van Albada and Lagendijk, 1985; Wolf and Maret, 1985) and brought about new measurement techniques, such as diffusive wave spectroscopy (Sheng, 1995) and coda wave interferometry (Snieder *et al*, 2002). Recently, attention has been paid to more complicated interactions of waves and scatterers (Wiersma *et al*, 1997; Lagendijk and Van Tiggelen, 1996), especially recurrent scattering (Podolsky and Lisyansky, 1997; Samelsohn and Mazar, 1997).

Recurrent scattering events take place when wave energy bounces off a particular scatterer, proceeds to bounce off at least one other scatterer, and subsequently returns to the original scatterer. This process forms a loop in the wave path. Van Tiggelen *et al* (1994) have made the connection between the simplest type of recurrent scattering and induced dipole-dipole coupling in atomic physics. Recurrent scattering has also been shown to decrease the magnitude of coherent backscattering (Wiersma, 1995b). Wiersma *et al* (1995a), moreover, have claimed that recurrent scattering could be involved in the Anderson localization of light.

In view of these far-reaching implications, it is remarkable that standard energy transport theory (or radiative transfer) ignores the contribution of recurrent scattering events by accounting only for self-avoiding scattering paths. This omission follows from the two approximations needed to render the complete multiple scattering problem at least tractable - the independent scattering approximation for the coherent beam, and the Boltzmann approximation for the diffuse intensity (Van Tiggelen and Maynard, 1998). So ubiquitous are these approximations in the literature that they are frequently invoked without justification. The redeeming aspect of these simplifications lies in their ability to conserve energy (Lagendijk and Van Tiggelen, 1996). The approximations have nothing to do with the interference effects that are neglected in radiative transfer. Once made, the approximations limit the multiple scattering to self-avoiding paths only.

For a finite number of scatterers, the self-avoiding assumption must break down for late times. After a time equal to the number of scatterers multiplied by the average time between successive scattering events, τ_{tr}, a wave must have revisited a scattering site on average at least once. Therefore, the predictions of radiative transfer, and the diffusion approximation (Morse and Feshbach, 1953), should, over time, progressively underestimate the amount of energy emerging from a scattering medium of finite extent.

In this paper, we represent multiple scattering as a summation over all possible paths, including recurrent paths, and quantify the underestimation of the intensity by the diffusion approximation in 2D. We do this by counting the self-avoiding and recurrent paths and comparing the relative numbers of each at different orders of scattering, i.e. once-scattered, twice-scattered, thrice-scattered, etc. We test this theory with an exact numerical solution of multiple scattering for 300 isotropic point scatterers based on a numerical implementation of Foldy's method (Foldy, 1945; Groenenboom and Snieder, 1995; Snieder and Scales, 1998). Because of recurrent scattering, the results show greater intensity than predicted by the diffusion approximation at late times. After independently measuring values of the average distance and average velocity between successive scattering events, ℓ_{tr} and v_E respectively, we show that this observed enhancement agrees well with the theory.

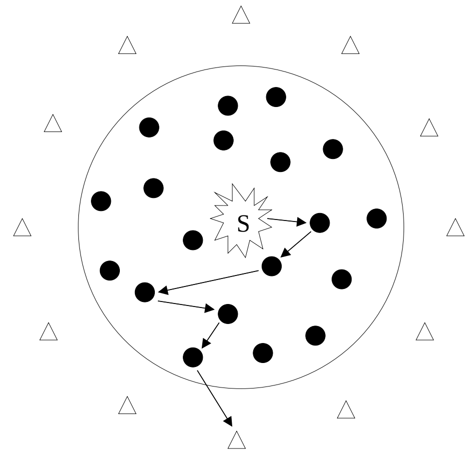

Figure 2. The setup for the numerical experiment: receivers are shown as triangles, scatterers as black spheres, and the source as S. The interior of the scattering region contains 300 scatterers. Dimensions are given in Table I.

A Numerical Scattering Experiment

The numerical experiment is depicted in Fig. 2, and the numerical values are listed in Table I. A ring of receivers, shown as triangles, encircles the scattering region. At the center of the scattering region, a point source (S) emits a band limited pulse centered about $t = 0$. Within the diffusion approximation, the Green's function for the average intensity, $I(r, r', t)$, is the solution of the diffusion equation for a point source:

TABLE I. Values of parameters for the numerical experiment and experimentally measured quantities

Quantity	Value
Radius of scattering region, R	5 mm
Radius to the arc of receivers	6 mm
Dominant wavelength, λ	2.5 mm
Background velocity, c_o	1500 mm/ms
Number of scatterers	300
Mean free path, ℓ_{tr}	1.41 ± .1 mm
Energy velocity, v_E	417 ± 36 mm/ms

$$D\nabla^2 I(r,r',t) - \frac{\partial I(r,r',t)}{\partial t} = \delta(r-r')\delta(t) \quad (1)$$

where $D = \frac{1}{2} v_E \ell_{tr}$ is the diffusion constant. The boundary conditions require that the solution is finite at the center of the scattering region (for physical reasons), and that it vanishes a distance $\frac{\pi \ell_{tr}}{4}$ outside:

$$I(r = R + \frac{\pi \ell_{tr}}{4}, r' = 0, t) = 0 \quad (2)$$

where R stands for the radius of the scattering region, and $r' = 0$ means the source is at the center. Equation (2) comes from the fact that no energy flows into the scattering region (Morse and Feshbach, 1953). The boundary condition cancels all incoming energy at the boundary on average. For a finite-sized scattering medium, this is the best that can be done within the diffusion approximation.

Supplemented with these boundary conditions, the solution of equation (1) can be expressed as an infinite series:

$$I(r, r' = 0, t) = \sum_{m=1}^{\infty} \frac{exp\left[-\left(\frac{z_{m,0}}{R+\frac{\pi \ell_{tr}}{4}}\right)^2 Dt\right] J_0\left(\frac{z_{m,0}\, r}{R+\frac{\pi \ell_{tr}}{4}}\right)}{J_1\left(z_{m,0}\right)^2} \quad (3)$$

where the Bessel functions of order zero and one are shown as J_0 and J_1, respectively. The terms $z_{m,0}$ denote the mth zero of the Bessel function of order zero.

We ignored absorption and assumed isotropic scattering from identical scatterers. The optical theorem (Snieder, 1999) was imposed for each scatterer so that energy was conserved. For simplicity, the scattering amplitude was independent of frequency. This avoided additional complications such as resonant multiple scattering (Nieuwenhuizen *et al*, 1992).

In order to characterize the transport of energy, ℓ_{tr} and v_E needed to be measured independently. By squaring the average wavefield at the receivers and correcting for cylindrical divergence, the maximum coherent intensity could be calculated. Performing this for several sizes of the scattering region while keeping the scatterer density constant (Tourin, 1999) provided the estimate of $\ell_{tr} = 1.41 \pm .1$ mm. This value for ℓ_{tr} is much larger than the value (.33 mm) obtained assuming self-avoiding paths (Groenenboom and Snieder, 1995; Tourin *et al*, 2000). The energy velocity was estimated by monitoring the arrival time of the maximum of the intensity with distance. After averaging this velocity over several realizations, we find $v_E = 417 \pm 36$ mm/ms.

The total intensity was computed by averaging the intensity at 12 receiver locations over 20 different realizations of the random medium. In a single realization, the receivers were sufficiently far apart to yield uncorrelated signals. Since source and receivers did not coincide for this experiment, coherent backscattering did not influence the measured intensity.

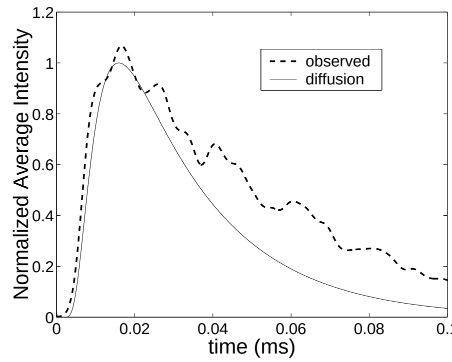

Figure 3. The measured total intensity and the diffusion approximation

By inserting the estimated values for ℓ_{tr} and v_E into equation (3), the accuracy of the self-avoiding assumption can be evaluated. As seen in Fig. 3, the solution of the diffusion equation fits the earliest arriving energy well; however, the growing number of recurrent paths with time eventually causes the diffusion approximation to underpredict the transmitted energy. The agreement between the theory and experiment in the initial part of the signal shows that the coherent beam has been strongly "attenuated" by the multiple scattering at this distance from the source (Paasschens, 1997). Dividing the measured total intensity by the diffusion solution yields the measured enhancement of energy due to recurrent scattering, shown in Fig. 4. The recurrent paths cause the enhancement to grow in time. Given the poor fit of the diffusion approximation to the measured total intensity in Fig. 3 for late times, a quantitative explanation of the enhancement, taking into account the self-avoiding assumption, is necessary.

Incorporating the Loop Paths

The enhancement can be explained with an argument based on counting the number of different paths through the scattering region. The paths that contribute most to the observed intensity for a certain number of self-intersections are taken into account. The number of self-intersections is equivalent to the number of times a wave revisits any scatterer. For zero self-intersections, there is only one type of path, the self-avoiding. Two distinct types show up for paths with one self-intersection: loops and folds (Wiersma, 1995b).

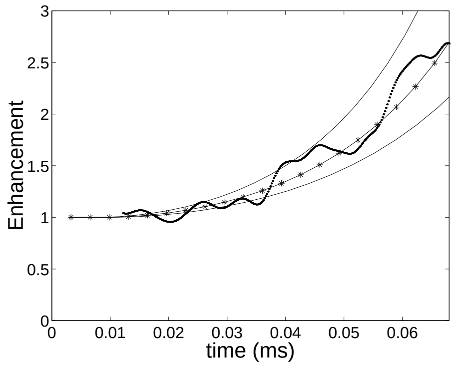

Figure 4. The measured enhancement and the theoretical prediction. The theoretical curve has dots along it, indicating time intervals of τ_{tr} or, equivalently, different orders of scattering. Error bars, calculated from the estimates of ℓ_{tr} and v_E, bracket the theoretical curve.

Folds occur when a wave, after encountering a scatterer, bounces off only one other scatterer before returning to the original scatterer. These paths describe a reverberation between a pair of scatterers. Loops occur when a wave bounces off more than one scatterer before returning to the original scatterer. These paths form a topological "loop".

Compared to the folds, loop paths are not as restricted. Loops can bounce off any number of scatterers before returning to the original scatterer, whereas the folds can only bounce off one. Hence, with increasing the number of scattering events, the number of loops increases much more rapidly than the folds and eventually dominates. This argument can be extended to paths with greater numbers of self-intersection. As described below, the small contribution of the folds is compounded by constructive interference that acts to amplify the loops. For these reasons, when enumerating the possible scattering paths (see appendix), we ignore folded scattering events. All orders of self-avoiding and loop recurrent paths are counted. The inclusion of all orders of recurrent paths is consistent with the expectations of the random walk in 2D (Samelsohn and Mazar, 1997).

A heuristic picture of the self-avoiding paths is shown in Fig. 5. The two arrows along the scattering path represent the wavefield squared (the intensity). Also shown is the diagrammatic representation of this particular path, a ladder diagram. In a diagram, the scatterers are represented by the symbol **X**. The ladder diagram defines the energy of a scattering path as the square of the wavefield that propagated along the path. Interaction between different scattering paths is not included. Compare this with Fig. 6, which shows energy making one loop around a recurrent path. The arch in the diagram connects identical scatterers.

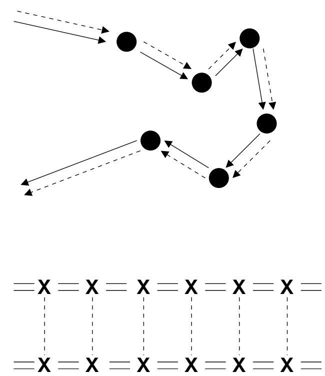

Figure 5. A self-avoiding path and its diagrammatic form, a ladder

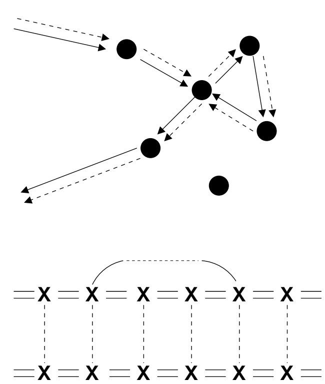

Figure 6. A loop path and one of its diagrams

Even though recurrent scattering is not the result of interference, the contribution of loop recurrent events to the intensity is amplified by interference with a factor of 2 per loop (Van Tiggelen *et al*, 1990). Fig. 7 shows two different paths of equal length traversing a loop. The two distinct scattering paths traverse the loop in different directions, as seen from their "most-crossed" diagram. Since the two paths have equal length, they

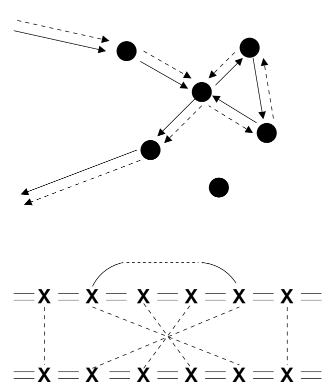

Figure 7. The "most-crossed" diagram for a particular loop path

constructively interfere with each other. Taking this and Fig. 6 into account, it is evident that all single loop paths have their intensity doubled by interference, in a way similar to coherent backscattering. Paths with two loops have their intensity quadrupled. The complete solution of the multiple scattering problem includes all the scattering paths and the interference between them. Only self-avoiding scattering paths are accounted for within the diffusion approximation. The enhancement of multiple wave scattering over the diffusion approximation for scattering of order n is thus given by*:

$$E(n) = \frac{(\text{\#self-avoiding}) + 2(\text{\#1loop}) + 4(\text{\#2loops}) \cdots}{(\text{\#self-avoiding})} \quad (4)$$

We assume that the transport mean free time, τ_{tr}, between successive scatterings is constant over all orders. The number of scatterers encountered after a time t is on average equal to $\frac{t}{\tau_{tr}}$. Therefore, the enhancement in equation (4) can be calculated at times $t = n\tau_{tr}$. Once τ_{tr} is measured independently, all that remains is the counting of the different types of paths. This can be done with the recursive algorithm described in the appendix.

Results

Using the independently measured values for ℓ_{tr} and v_E to obtain τ_{tr} $(=\frac{\ell_{tr}}{v_E})$, the theoretical prediction of equation (4) can be compared to the measured enhancement of the intensity from the numerical experiment over the diffusion approximation. The two curves are plotted in Fig. 4. Agreement is seen over 21 orders of scattering. This confirms that, although the diffusion approximation grossly underestimated the measured intensity, the parameters ℓ_{tr} and v_E are correct since they give a consistent estimate of the enhancement once recurrent scattering paths are incorporated.

Past 21 orders of scattering, the theoretical enhancement overestimates the measured enhancement. A number of factors could be responsible for this. The theory predicts an enhancement factor of ∞ at times past $N\tau_{tr}$, since the number of self-avoiding paths, the denonimator of equation (4), is zero. The diffusion approximation solution, equation (3), as a summation of decaying exponentials, cannot vanish at any time; therefore, the measured enhancement can never become infinite, as the theory expects. One possibility to improve the theory is to make τ_{tr} a distribution, instead of simply an average quantity. Such a procedure would be in line with P. W. Anderson's opinion on the main contribution of localization theory to the study of random media:

> "What we really need to know is the ***probability distribution*** ... ***not*** its average, because it's only each specific instance we are interested in. I would like to emphasize that this is the important, and deeply new, step taken here: the willingness to deal with ***distributions***, not ***averages***." (Anderson, 1978)

* It is interesting to note that, without interference effects, the multiples of 2 in equation (4) would all be equal to one; however, in this case, an enhancement still occurs. Hence, recurrent scattering is an example of a multiple scattering phenomenon that should appear in experiments even when reciprocity is broken by the application of an external magnetic field (Van Tiggelen and Maynard, 1998). The presence of recurrent scattering without reciprocity is supported by the recently accepted view that localization is not destroyed, and is at most modified, by the absence of time-reversal symmetry (Van Tiggelen, 1999).

Conclusions

In developing the self-consistent theory of localization, Vollhardt and Wolfle (1992) interpreted recurrent paths as a mechanism for slowing down of diffusion. We have shown that in 2D the recurrent paths lead to a breakdown of the diffusion model (Abrahams, 1979; Van Tiggelen, 1999). Additionally, we have proposed and tested a theory that details exactly how the diffusion model fails for late times.

An important step for the future would be the extension of this theory to 3D. Recurrent scattering events always occur in 2D for any amount of disorder since 2D is the lower critical dimension for localization. For a finite scattering volume in 3D, it is not obvious how the effects of recurrent scattering will depend on $k\ell_{tr} = \frac{2\pi}{\lambda}\ell_{tr}$, the measure of scattering strength.

Acknowledgements

We thank John A. Scales for engendering interest in multiple scattering and coherent backscattering and for many discussions. He is also responsible for the code used to compute the energy velocity.

APPENDIX

Let N equal the number of scatterers and n equal the order of scattering (number of scatterings). The first loop shows up at 4 scattering events. After that, new orders of loops show up every 3 additional scatterings (n = 7,10,13,16,...). In other words, beginning at the seventh scattering event, paths with two loops become possible. Paths with three loops become possible at ten scattering events. A table of the enhancement of the numerically

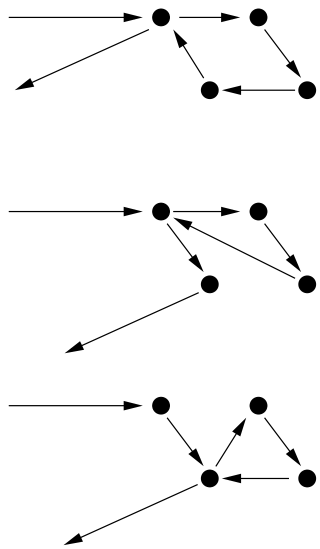

Figure 8. The three types of one loop paths at five scatterings (n=5)

measured total intensity over the diffusion solution for the first 9 scattering events can be expanded out in its component parts:

n	$E(t = n\tau_{tr})$
1	1
2	1
3	1
4	$1 + 2{\cdot}1{\cdot}(\frac{1}{(N-3)})$
5	$1 + 2{\cdot}3{\cdot}(\frac{1}{(N-4)})$
6	$1 + 2{\cdot}6{\cdot}(\frac{1}{(N-5)})$
7	$1 + 2{\cdot}10{\cdot}(\frac{1}{(N-6)}) + 4{\cdot}3{\cdot}(\frac{1}{(N-5)(N-6)})$
8	$1 + 2{\cdot}15{\cdot}(\frac{1}{(N-7)}) + 4{\cdot}15{\cdot}(\frac{1}{(N-6)(N-7)})$
9	$1 + 2{\cdot}21{\cdot}(\frac{1}{(N-8)}) + 4{\cdot}45{\cdot}(\frac{1}{(N-7)(N-8)})$
⋮	⋮

At five scatterings (n=5), the enhancement is one plus the multiplication of three terms. The product of these terms is the enhancement from the paths with one loop. The first term, 2, represents the doubling of the intensity due to the constructive interference of the two ways around one loop. The second term, 3, is the number of different types of one loop paths that can be formed at five scatterings. The three types are illustrated in Fig. 8. The third term counts the relative number of one loop paths compared to the self-avoiding paths. It depends on the total number of scatterers, N. At five scatterings, the number of self-avoiding paths is equal to $\frac{N!}{(N-5)!}$. The number of one loop paths is given by $\frac{N!}{(N-4)!}$. Dividing the number of one loop paths by the number of self-avoiding paths yields the third term, $\frac{1}{(N-4)}$.

We will call the first term the interference term, the second term the path term, and the third term the counting term. It turns out that the hardest of these terms to obtain is the path term. For the one loop paths, the interference term doubles the contribution for all orders of scattering, n. For $n \geq 4$, the path term of the one loops paths equals $\frac{1}{2} * (n-3) * (n-2)$. The counting term becomes larger according to the combinatorics as $\frac{1}{(N-n+1)}$. From this expression, one can see that the predicted enhancement becomes infinite for $n > N$.

The two loop paths begin at seven scatterings (n=7). Their interference term is four since there are four ways around a path with two loops. The path term for the two loop paths can be defined recursively. At n=7, the path term for the two loop paths is equal to its value at the previous scattering order (zero in this case), plus the quantity $(1 + 2^{(\#\text{ of loops})-1} + (n - (3 * \#\text{ of loops} + 1)))$ times the path term for the one loop paths at three scatterings previous (n=4). For two loop paths at n=7, $(1 + 2^{(\#\text{ of loops})-1} + (n - (3 * \#\text{ of loops} + 1)))$ is equal to 3. The path term for the one loop paths at n=4 is 1. Therefore, with this recursive definition, the path term for the two loop paths at n=7 is equal to $0 + (3 * 1)$, or 3, as shown in the table. Accordingly, the path term for two loop paths at n=8 is equal to its value at the previous scattering order (we just showed it equal to three), plus the quantity $(1 + 2^{(\#\text{ of loops})-1} + (n - (3 * \#\text{ of loops} + 1)))$ times the path term for the one loop paths at three scatterings previous (now n=5). Refering to the table, the path term for two loop paths at n=8 is $3 + (4 * 3)$, or 15. At n=9, it is equal to $15 + (5 * 6)$, or 45. The counting term for the two loop paths follows from the combinatorics. For example, at 7 scatterings, the number of self-avoiding paths is equal to $\frac{N!}{(N-7)!}$. The number of two loop paths is given by $\frac{N!}{(N-5)!}$. Dividing the number of two loop paths by the number of self-avoiding paths yields the counting term, $\frac{1}{(N-5)(N-6)}$.

The three loop paths (not shown in table) begin to become possible at n=10. Because of the eight ways around three loops, the interference term for the three loop paths is eight. The path term for the three loop paths at n=10 follows the same recursive form as for the two loop paths. At n=10, the path term is equal to its value at the previous scattering order (zero in this case), plus the quantity $(1 + 2^{(\#\text{ of loops})-1} + (n - (3 * \#\text{ of loops} + 1)))$ times the path term for the two loop paths at three scatterings previous (n=7). For the three loop paths at n=10, the quantity in parenthesis is equal to 5. The path term for the two loop paths at three scatterings previous is 3. The requirement of three scatterings previous never changes for all orders of loops. Hence the path term for the three loop paths at n=10 is equal to $0 + (5 * 3)$, or 15. Notice that the recursion for the path term of three loop paths is only dependent on the path term of the two loop paths, just as the path term of two loop paths was only dependent on the path term of the one loop paths. This allows a compact recursive algorithm to take care of the whole series of loops.

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