Plane-wave attenuation anisotropy in orthorhombic media

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ABSTRACT

Orthorhombic velocity and attenuation models are needed in the interpretation of the azimuthal variation of seismic signatures recorded over fractured reservoirs. Here, we develop an analytic framework for describing the attenuation coefficients in orthorhombic media with orthorhombic attenuation (i.e., the symmetry of both the real and imaginary parts of the stiffness tensor is orthorhombic).

The analogous form of the Christoffel equation in the symmetry planes of orthorhombic and VTI (transversely isotropic with a vertical symmetry axis) media helps to obtain the symmetry-plane attenuation coefficients by adapting the existing VTI equations. To take full advantage of this equivalence with transverse isotropy, we introduce a set of attenuation-anisotropy parameters similar to the VTI parameters ϵ_Q, δ_Q, and γ_Q. This notation, based on the same principle as Tsvankin's velocity-anisotropy parameters for orthorhombic media, leads to simple linearized equations for the symmetry-plane attenuation coefficients of all three modes (P, S_1, and S_2).

The attenuation-anisotropy parameters also allow us to simplify the P-wave attenuation coefficient $\mathcal{A}_P$ outside the symmetry planes under the assumption of weak attenuation and weak velocity and attenuation anisotropy. The approximate $\mathcal{A}_P$ has the same form as the linearized phase-velocity function, with Tsvankin's velocity parameters $\epsilon^{(1,2)}$ and $\delta^{(1,2,3)}$ replaced by the attenuation parameters $\epsilon_Q^{(1,2)}$ and $\delta_Q^{(1,2,3)}$. The exact attenuation coefficient $\mathcal{A}_P$, however, also depends on the velocity-anisotropy parameters, while the body-wave velocities are almost unperturbed by the presence of attenuation.

The reduction in the number of parameters responsible for the P-wave attenuation and the simple approximation for the coefficient $\mathcal{A}_P$ provide a basis for inverting P-wave attenuation measurements from orthorhombic media. The attenuation processing has to be preceded by anisotropic velocity analysis that can be performed (in the absence of pronounced velocity dispersion) using existing algorithms for nonattenuative media.

Key words: attenuation, orthorhombic symmetry, anisotropy parameters, plane waves, linearized approximation

1 INTRODUCTION

Effective velocity models of fractured reservoirs often have orthorhombic or an even lower symmetry (Schoenberg and Helbig, 1997; Bakulin et al., 2000). It is likely that polar and azimuthal velocity variations in orthorhombic formations are accompanied by directionally dependent attenuation. Indeed, systems of aligned fractures or pores are among the most common physical reasons for anisotropic attenuation (e.g., Mavko and Nur, 1979; Akbar, 1993; Pointer et al., 1996).

Physical modeling shows that the P-wave attenuation coefficient in the direction perpendicular to aligned pores or fractures is higher than that parallel to the pores (Akbar, 1993). Similar results were obtained by Zhu and Tsvankin (2005b) for a synthetic material, in which thin layers of paper bonded with phenolic resin (i.e., aligned heterogeneities) create extremely strong attenuation anisotropy. Pointer et al. (1996) found that significant dissipation of energy is caused by the movement of fluids in interconnected pathways (crack-pore networks). The relationship between the azimuthal variation of attenuation and horizontal permeability measured over a fractured reservoir was discussed by Lynn et al. (1999). On the whole, existing experimental data indicate that both velocity and attenuation in fractured rocks vary with angle, with the type and magnitude of the anisotropy controlled by such factors as the shape, distribution, and orientation of aligned fractures and pores.

When the dominant wavelength is much larger than the characteristic size of heterogeneities, the scattering phenomena can be ignored, and the medium can be treated as effectively homogeneous. This paper is devoted to the attenuation of plane waves propagating in a homogeneous medium that has orthorhombic symmetry for both the velocity function and attenuation coefficient. As in our previous work on transversely isotropic (TI) media (Zhu and Tsvankin, 2004, 2005a,b), we study the *normalized* attenuation coefficient defined as

$$\mathcal{A} \equiv \frac{k^I}{k}\,, \tag{1}$$

where k and k^I are the real and the imaginary parts of the wavenumber, respectively. The coefficient $\mathcal{A}$ determines the rate of the amplitude decay per wavelength. The two main assumptions used here to simplify the analytic description of attenuation are as follows:
1) Wave propagation is "homogeneous," which means that the real and the imaginary parts of the wave vector are parallel to each other ($\mathbf{k}^I \parallel \mathbf{k}$).
2) The symmetry of the imaginary part of the stiffness matrix (or stiffness tensor) coincides with that of the real part. This assumption ensures that the quality-factor matrix $\mathbf{Q}$ (Carcione, 2001; Zhu and Tsvankin, 2004, 2005a) has the same structure as the real part of the stiffness matrix that governs the velocity anisotropy.

The main challenge in describing the attenuation anisotropy in orthorhombic materials is in the large number of parameters that control the attenuation coefficients. Because of the coupling between the velocity and attenuation anisotropy, the coefficient $\mathcal{A}$ depends (for a fixed orientation of the symmetry planes) on the nine real stiffness coefficients and nine elements of the quality matrix. Here, we show that significant simplifications can be achieved by extending the principle of Tsvankin's (1997, 2001) notation for velocity anisotropy to attenuative orthorhombic media.

The equivalence between the complex Christoffel equation in the symmetry planes of orthorhombic and VTI (TI with a vertical symmetry axis) media makes it possible to obtain the symmetry-plane attenuation coefficients from the corresponding VTI equations. As shown by Zhu and Tsvankin (2004, 2005a), attenuation anisotropy in VTI media can be conveniently described by the Thomsen-style parameters ϵ_Q, δ_Q, and γ_Q. Adapting the results of Zhu and Tsvankin (2004, 2005a) for the symmetry planes of orthorhombic media, we introduce a set of seven anisotropy parameters that fully characterizes (in combination with the velocity parameters) directionally-dependent attenuation in orthorhombic materials. Linearizing the P-wave attenuation coefficient in the limit of weak attenuation and weak anisotropy yields a simple expression outside the symmetry planes that has the same form as Tsvankin's (1997, 2001) weak-anisotropy approximation for the velocity function. The accuracy of the approximate attenuation coefficient is verified using numerical tests for models with substantial attenuation and velocity anisotropy.

2 CHRISTOFFEL EQUATION FOR ATTENUATIVE ORTHORHOMBIC MEDIA

A harmonic plane wave propagating in an attenuative medium has the form

$$\tilde{\mathbf{u}} = \tilde{\mathbf{U}} \exp\left[i(\omega t - \tilde{\mathbf{k}}\mathbf{x})\right], \tag{2}$$

where $\tilde{\mathbf{U}}$ is the polarization vector, and $\tilde{\mathbf{k}} = \mathbf{k} - i\mathbf{k}^I$ is the wave vector (both vectors are complex). We consider plane-wave propagation in orthorhombic media with orthorhombic attenuation, which means that the symmetry of the imaginary part of the stiffness matrix is identical to that of the real part. Then it is convenient to choose a Cartesian coordinate system aligned with the "natural" coordinate frame of the model, so that each coordinate plane coincides with one of the symmetry planes.

Substituting the plane wave (2) into the wave equation yields the following Christoffel equation:

$$\begin{bmatrix} \tilde{c}_{11}\tilde{k}_1^2 + \tilde{c}_{66}\tilde{k}_2^2 + \tilde{c}_{55}\tilde{k}_3^2 - \rho\omega^2 & (\tilde{c}_{12}+\tilde{c}_{66})\tilde{k}_1\tilde{k}_2 & (\tilde{c}_{13}+\tilde{c}_{55})\tilde{k}_1\tilde{k}_3 \\ (\tilde{c}_{12}+\tilde{c}_{66})\tilde{k}_1\tilde{k}_2 & \tilde{c}_{66}\tilde{k}_1^2 + \tilde{c}_{22}\tilde{k}_2^2 + \tilde{c}_{44}\tilde{k}_3^2 - \rho\omega^2 & (\tilde{c}_{23}+\tilde{c}_{44})\tilde{k}_2\tilde{k}_3 \\ (\tilde{c}_{13}+\tilde{c}_{55})\tilde{k}_1\tilde{k}_3 & (\tilde{c}_{23}+\tilde{c}_{44})\tilde{k}_2\tilde{k}_3 & \tilde{c}_{55}\tilde{k}_1^2 + \tilde{c}_{44}\tilde{k}_2^2 + \tilde{c}_{33}\tilde{k}_3^2 - \rho\omega^2 \end{bmatrix} \cdot \begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{Bmatrix} = 0\,, \tag{3}$$

where $\tilde{c}_{ij} = c_{ij} + ic^I_{ij}$ is the complex stiffness coefficient. Following Carcione (2001) and Zhu and Tsvankin (2004), the elements of the quality-factor matrix can be defined as (no index summation is implied)

$$Q_{ij} \equiv \frac{c_{ij}}{c^I_{ij}} \,. \tag{4}$$

Assuming homogeneous wave propagation ($\mathbf{k} \parallel \mathbf{k}^I$), the wave vector can be expressed through the unit vector $\mathbf{n}$ in the slowness direction: $\tilde{\mathbf{k}} = \mathbf{n}\,\tilde{k}$. Then the Christoffel equation (3) becomes

$$\left[\tilde{G}_{ij} - \rho\tilde{V}^2\delta_{ik}\right]\tilde{U}_k = 0\,, \tag{5}$$

where $\tilde{V} = \dfrac{\omega}{\tilde{k}}$ is the complex phase velocity, and $\tilde{G}_{ij}$ are the elements of the complex Christoffel matrix:

$$\begin{aligned}
\tilde{G}_{11} &= \tilde{c}_{11}n_1^2 + \tilde{c}_{66}n_2^2 + \tilde{c}_{55}n_3^2\,,\\
\tilde{G}_{22} &= \tilde{c}_{66}n_1^2 + \tilde{c}_{22}n_2^2 + \tilde{c}_{44}n_3^2\,,\\
\tilde{G}_{33} &= \tilde{c}_{55}n_1^2 + \tilde{c}_{44}n_2^2 + \tilde{c}_{33}n_3^2\,,\\
\tilde{G}_{12} &= (\tilde{c}_{12} + \tilde{c}_{66})n_1n_2\,,\\
\tilde{G}_{13} &= (\tilde{c}_{13} + \tilde{c}_{55})n_1n_3\,,\\
\tilde{G}_{23} &= (\tilde{c}_{23} + \tilde{c}_{44})n_2n_3\,.
\end{aligned} \tag{6}$$

The components of the unit slowness (phase) vector $\mathbf{n}$ can be expressed through the polar phase angle θ and the azimuthal phase angle ϕ: $n_1 = \sin\theta\cos\phi$, $n_2 = \sin\theta\sin\phi$, $n_3 = \cos\theta$.

Note that although equation (5) has the same form as the Christoffel equation in nonattenuative (purely elastic) orthorhombic media, the velocity, polarization and stiffnesses are complex. As a result, plane-wave propagation in attenuative media is described by two coupled equations obtained by separating the real and imaginary parts of the Christoffel equation.

3 ATTENUATION COEFFICIENTS IN THE SYMMETRY PLANES

Suppose that the plane wave (2) propagates in the $[x_1, x_3]$-plane, so $n_1 = \sin\theta$, $n_2 = 0$, and $n_3 = \cos\theta$. Then the Christoffel equation (3) simplifies to

$$\begin{bmatrix} \tilde{c}_{11}\tilde{k}_1^2 + \tilde{c}_{55}\tilde{k}_3^2 - \rho\omega^2 & 0 & (\tilde{c}_{13} + \tilde{c}_{55})\tilde{k}_1\tilde{k}_3 \\ 0 & \tilde{c}_{66}\tilde{k}_1^2 + \tilde{c}_{44}\tilde{k}_3^2 - \rho\omega^2 & 0 \\ (\tilde{c}_{13} + \tilde{c}_{55})\tilde{k}_1\tilde{k}_3 & 0 & \tilde{c}_{55}\tilde{k}_1^2 + \tilde{c}_{33}\tilde{k}_3^2 - \rho\omega^2 \end{bmatrix} \begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{Bmatrix} = 0\,. \tag{7}$$

Equation (7) has the same form as the Christoffel equation for VTI media with VTI attenuation discussed in detail by Zhu and Tsvankin (2004, 2005a). The only difference between the two equations is that while for VTI media $\tilde{c}_{44} = \tilde{c}_{55}$, that is generally not the case for orthorhombic symmetry. However, the stiffness $\tilde{c}_{44}$ influences only the SH-wave polarized perpendicular to the propagation plane (see below),while $\tilde{c}_{55}$ contributes to the velocity and attenuation of the in-plane polarized waves (P and SV). Therefore, the well-known equivalence between the Christoffel equation in purely elastic VTI media and symmetry planes of orthorhombic media (e.g., Tsvankin, 1997, 2001) holds for attenuative models with the same symmetries of the real and imaginary parts of stiffness tensor.

Since the Christoffel matrix for wave propagation in the $[x_1, x_3]$-plane has four vanishing elements, equation (7) splits into two separate equations, one for the SH-wave polarized in the x_2-direction (the displacement component $\tilde{U}_2$), and the other for the in-plane polarized P- and SV-waves (the components $\tilde{U}_1$ and $\tilde{U}_3$). The solutions for the velocity and attenuation of all three modes can be obtained by simply adapting the results of Zhu and Tsvankin (2004, 2005a) for VTI media.

Assuming homogeneous wave propagation ($\tilde{\mathbf{k}} = \mathbf{n}\,\tilde{k}$), the Christoffel equation for the SH-wave takes the form

$$\left(\tilde{c}_{66}\sin^2\theta + \tilde{c}_{44}\cos^2\theta\right)\tilde{k}^2 - \rho\omega^2 = 0\,. \tag{8}$$

Using the VTI result of Zhu and Tsvankin (2004, 2005a), the normalized attenuation coefficient of SH-waves in the $[x_1, x_3]$-plane can be obtained from equation (8) as

$$\mathcal{A}^{(2)}_{SH} = \sqrt{1 + (Q_{44}\,\alpha^{(2)})^2} - Q_{44}\,\alpha^{(2)}\,, \tag{9}$$

where the superscript (2) stands for the x_2-axis orthogonal to the propagation plane (the same convention as in

Tsvankin, 1997, 2001), and

$$\alpha^{(2)} \equiv \frac{(1+2\gamma^{(2)})\sin^2\theta + \cos^2\theta}{(1+2\gamma^{(2)})\dfrac{Q_{44}}{Q_{66}}\sin^2\theta + \cos^2\theta}\,.$$

For P- and SV-waves in the regime of homogeneous wave propagation, equation (7) reduces to

$$\begin{bmatrix} \left(\tilde{c}_{11}\sin^2\theta + \tilde{c}_{55}\cos^2\theta\right)\tilde{k}^2 - \rho\omega^2 & (\tilde{c}_{13}+\tilde{c}_{55})\sin\theta\cos\theta\,\tilde{k}^2 \\ (\tilde{c}_{13}+\tilde{c}_{55})\sin\theta\cos\theta\,\tilde{k}^2 & \left(\tilde{c}_{55}\sin^2\theta + \tilde{c}_{33}\cos^2\theta\right)\tilde{k}^2 - \rho\omega^2 \end{bmatrix} \begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_3 \end{Bmatrix} = 0\,. \tag{10}$$

The wavenumber obtained from equation (10) is described by an expression analogous to that in nonattenuative VTI media (e.g., Tsvankin, 2001):

$$\tilde{k} = \omega\sqrt{2\rho}\,\Big\{(\tilde{c}_{11}+\tilde{c}_{55})\sin^2\theta + (\tilde{c}_{33}+\tilde{c}_{55})\cos^2\theta \pm \sqrt{\left[(\tilde{c}_{11}-\tilde{c}_{55})\sin^2\theta - (\tilde{c}_{33}-\tilde{c}_{55})\cos^2\theta\right]^2 + 4\,(\tilde{c}_{13}+\tilde{c}_{55})^2\sin^2\theta\cos^2\theta}\Big\}^{-1/2}\,. \tag{11}$$

The normalized attenuation coefficients $\mathcal{A}^{(2)}_{P,SV}$ were derived from the complex part of equation (11) by Zhu and Tsvankin (2004, 2005a). For example, the P-wave coefficients $\mathcal{A}^{(2)}_P$ in the vertical and horizontal directions are given by

$$\mathcal{A}^{(2)}_P(\theta=0^\circ) = Q_{33}\left(\sqrt{1+1/Q_{33}^2} - 1\right) \approx \frac{1}{2Q_{33}}\,, \tag{12}$$

$$\mathcal{A}^{(2)}_P(\theta=90^\circ) = Q_{11}\left(\sqrt{1+1/Q_{11}^2} - 1\right) \approx \frac{1}{2Q_{11}}\,. \tag{13}$$

The SV-wave attenuation coefficient in both the vertical and horizontal directions is

$$\mathcal{A}^{(2)}_{SV}(\theta=0^\circ) = \mathcal{A}^{(2)}_{SV}(\theta=90^\circ) = Q_{55}\left(\sqrt{1+1/Q_{55}^2} - 1\right) \approx \frac{1}{2Q_{55}}\,. \tag{14}$$

For plane-wave propagation in the $[x_2, x_3]$-plane ($n_1 = 0$, $n_2 = \sin\theta$, and $n_3 = \cos\theta$), the Christoffel equation (3) gives

$$\begin{bmatrix} \tilde{c}_{66}\tilde{k}_1^2 + \tilde{c}_{55}\tilde{k}_3^2 - \rho\omega^2 & 0 & 0 \\ 0 & \tilde{c}_{22}\tilde{k}_1^2 + \tilde{c}_{44}\tilde{k}_3^2 - \rho\omega^2 & (\tilde{c}_{23}+\tilde{c}_{44})\tilde{k}_1\tilde{k}_3 \\ 0 & (\tilde{c}_{23}+\tilde{c}_{44})\tilde{k}_1\tilde{k}_3 & \tilde{c}_{44}\tilde{k}_1^2 + \tilde{c}_{33}\tilde{k}_3^2 - \rho\omega^2 \end{bmatrix} \begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{Bmatrix} = 0\,. \tag{15}$$

The SH-wave, which is polarized in the x_1-direction, is described by the element $\tilde{c}_{66}\tilde{k}_1^2 + \tilde{c}_{55}\tilde{k}_3^2 - \rho\omega^2$ of the matrix in equation (15). It is clear from equations (7) and (15) that both the velocity and attenuation of the SH-wave can be obtained from the corresponding equations for the $[x_1, x_3]$-plane (or VTI media) by making the substitution $4 \to 5$ in the subscripts of the stiffnesses and elements Q_{ij}. For example, the normalized attenuation coefficient for homogeneous SH-wave propagation can be adapted from equation (9):

$$\mathcal{A}^{(1)}_{SH} = \sqrt{1 + (Q_{55}\,\alpha^{(1)})^2} - Q_{55}\,\alpha^{(1)}\,, \tag{16}$$

where

$$\alpha^{(1)} \equiv \frac{(1+2\gamma^{(1)})\sin^2\theta + \cos^2\theta}{(1+2\gamma^{(1)})\dfrac{Q_{55}}{Q_{66}}\sin^2\theta + \cos^2\theta}\,.$$

The velocity and attenuation of P- and SV-waves in the $[x_2, x_3]$-plane are described by (for homogeneous wave propagation)

$$\begin{bmatrix} \left(\tilde{c}_{22}\sin^2\theta + \tilde{c}_{44}\cos^2\theta\right)\tilde{k}^2 - \rho\omega^2 & (\tilde{c}_{23}+\tilde{c}_{44})\sin\theta\cos\theta\,\tilde{k}^2 \\ (\tilde{c}_{23}+\tilde{c}_{44})\sin\theta\cos\theta\,\tilde{k}^2 & \left(\tilde{c}_{44}\sin^2\theta + \tilde{c}_{33}\cos^2\theta\right)\tilde{k}^2 - \rho\omega^2 \end{bmatrix} \begin{Bmatrix} \tilde{U}_2 \\ \tilde{U}_3 \end{Bmatrix} = 0\,. \tag{17}$$

The P- and SV-wave attenuation coefficients can be obtained from the equations for the $[x_1, x_3]$-plane using the following substitutions in the subscripts: $1 \to 2$ and $5 \to 4$.

The same substitutions were used by Tsvankin (1997, 2001) in his extension of the VTI velocity equations to the symmetry planes of orthorhombic media. The equivalence with vertical transverse isotropy is also valid for the complex Christoffel equation in the $[x_1, x_2]$ symmetry plane.

4 ATTENUATION-ANISOTROPY PARAMETERS

The Thomsen-style notation for velocity anisotropy suggested by Tsvankin (1997, 2001) helps to simplify the analytic description of a wide range of seismic signatures for orthorhombic media. Tsvankin's parameters provided a basis for developing efficient seismic inversion and processing methods operating with orthorhombic models (Grechka and Tsvankin, 1999; Grechka et al., 1999; Bakulin et al., 2000). Here, we extend his approach to attenuative orthorhombic media with the main goal of defining the parameter combinations that govern the directionally dependent attenuation coefficient.

Since our notation is designed primarily for reflection data, we choose the P- and S-wave attenuation coefficients in the vertical (x_3) direction ($\mathcal{A}_{P0}$ and $\mathcal{A}_{S0}$) as the reference isotropic quantities. The coefficient $\mathcal{A}_{S0}$ corresponds to the S-wave polarized in the x_1-direction, which may be either the fast or slow shear mode depending on the relationship between the stiffnesses c_{44} and c_{55}. According to equations (12) and (14), the approximate (accurate to the second order in $1/Q$) coefficients $\mathcal{A}_{P0}$ and $\mathcal{A}_{S0}$ are given by

$$\mathcal{A}_{P0} \equiv \frac{1}{2Q_{33}}\,, \tag{18}$$

$$\mathcal{A}_{S0} \equiv \frac{1}{2Q_{55}}\,. \tag{19}$$

To characterize the attenuation of waves propagating in the $[x_1, x_3]$-plane, we define three attenuation-anisotropy parameters analogous to the Thomsen-style parameters ϵ_Q, δ_Q, and γ_Q introduced for VTI media with VTI attenuation by Zhu and Tsvankin (2004, 2005a). The parameters $\epsilon_Q^{(2)}$ and $\gamma_Q^{(2)}$ (the superscript "(2)" stands for the x_2-axis perpendicular to the $[x_1, x_3]$-plane) determine the fractional difference between the normalized attenuation coefficients in the x_1- and x_3-directions for the P- and SH-waves, respectively. Another parameter, $\delta_Q^{(2)}$, is expressed through the second derivative of the P-wave attenuation coefficient in the vertical direction and, therefore, governs the P-wave attenuation for near-vertical propagation in the $[x_1, x_3]$-plane.

$$\epsilon_Q^{(2)} \equiv \frac{Q_{33}-Q_{11}}{Q_{11}}\,, \tag{20}$$

$$\delta_Q^{(2)} \equiv \frac{\dfrac{Q_{33}-Q_{55}}{Q_{55}}c_{55}\dfrac{(c_{13}+c_{33})^2}{(c_{33}-c_{55})}+2\dfrac{Q_{33}-Q_{13}}{Q_{13}}c_{13}(c_{13}+c_{55})}{c_{33}(c_{33}-c_{55})} \tag{21}$$

$$\approx 4\,\frac{Q_{33}-Q_{55}}{Q_{55}}\,g^{(2)}+2\,\frac{Q_{33}-Q_{13}}{Q_{13}}\,(1+2\delta^{(2)}-2g^{(2)})\,, \tag{22}$$

$$\gamma_Q^{(2)} \equiv \frac{Q_{44}-Q_{66}}{Q_{66}}\,, \tag{23}$$

where equation (22) for $\delta_Q^{(2)}$ is simplified by assuming that the ratio $g^{(2)} \equiv \dfrac{c_{55}}{c_{33}}$ and the absolute value of Tsvankin's velocity-anisotropy parameter $\delta^{(2)}$ are small. Since the Christoffel equation in the $[x_1, x_3]$-plane has the same form as in VTI media, equations (20)–(23) are identical to the definitions of the corresponding VTI parameters. In contrast to VTI models, however, the parameters of orthorhombic media with the subscripts "55" and "44" are generally different, and one needs to use c_{55} and Q_{55} (not c_{44} or Q_{44}) in equation (21) and Q_{44} (not Q_{55}) in equation (23).

Similarly, we adapt the VTI definitions of Zhu and Tsvankin (2004, 2005a) to introduce three attenuation-anisotropy parameters in the $[x_2, x_3]$-plane:

$$\epsilon_Q^{(1)} \equiv \frac{Q_{33}-Q_{22}}{Q_{22}}\,, \tag{24}$$

$$\delta_Q^{(1)} \equiv \frac{\dfrac{Q_{33}-Q_{44}}{Q_{44}}c_{44}\dfrac{(c_{23}+c_{33})^2}{(c_{33}-c_{44})}+2\dfrac{Q_{33}-Q_{23}}{Q_{23}}c_{23}(c_{23}+c_{44})}{c_{33}(c_{33}-c_{44})} \tag{25}$$

$$\approx 4\,\frac{Q_{33}-Q_{44}}{Q_{44}}\,g^{(1)}+2\,\frac{Q_{33}-Q_{23}}{Q_{23}}\,(1+2\delta^{(1)}-2g^{(1)})\,, \tag{26}$$

$$\gamma_Q^{(1)} \equiv \frac{Q_{55}-Q_{66}}{Q_{66}}\,. \tag{27}$$

In equation (26), $\delta^{(1)}$ is the velocity-anisotropy parameter defined in the $[x_2, x_3]$-plane (Tsvankin, 1997, 2001), and $g^{(1)} \equiv \dfrac{c_{44}}{c_{33}}$. Since the attenuation coefficient is supposed to be positive (otherwise, the amplitude will increase with distance), the diagonal components of the **Q** matrix have to be positive as well. This constraint implies the parameters $\epsilon_Q^{(1)}$, $\epsilon_Q^{(2)}$, $\gamma_Q^{(1)}$, and $\gamma_Q^{(2)}$ are always larger than -1.

The only component of the **Q**-matrix that is not involved in the definitions of the reference isotropic quantities and the attenuation-anisotropy parameters in the vertical symmetry planes is Q_{12}. Following the approach of Tsvankin (1997, 2001), we use Q_{12} to introduce one more anisotropy parameter, $\delta_Q^{(3)}$, which plays the role of the VTI parameter δ_Q in the $[x_1, x_2]$-plane (x_1 is treated as the symmetry axis of the equivalent VTI model):

$$\delta_Q^{(3)} \equiv \frac{\frac{Q_{11}-Q_{66}}{Q_{66}} c_{66} \frac{(c_{11}+c_{12})^2}{(c_{11}-c_{66})} + 2\frac{Q_{11}-Q_{12}}{Q_{12}} c_{12}(c_{12}+c_{66})}{c_{11}(c_{11}-c_{66})} \tag{28}$$

$$\approx 4\,\frac{Q_{11}-Q_{66}}{Q_{66}}\, g^{(3)} + 2\,\frac{Q_{11}-Q_{12}}{Q_{12}}\,(1+2\delta^{(3)}-2g^{(3)}), \tag{29}$$

where $\delta^{(3)}$ is another Tsvankin's velocity-anisotropy parameter defined in the $[x_1, x_2]$-plane, and $g^{(3)} \equiv \frac{c_{66}}{c_{11}}$. Although it is also possible to introduce the parameters $\epsilon_Q^{(3)}$ and $\gamma_Q^{(3)}$ in the $[x_1, x_2]$-plane, they would be redundant.

The nine attenuation-anisotropy parameters defined in equations (18)–(29), combined with Tsvankin's (1997, 2001) velocity-anisotropy parameters, are sufficient to fully characterize plane-wave attenuation in orthorhombic media. An additional practically important parameter responsible for the differential attenuation of the split S-waves in the vertical (x_3) direction is described in the next section.

5 APPROXIMATE ATTENUATION COEFFICIENTS IN THE SYMMETRY PLANES

The equivalence between plane-wave propagation in the symmetry planes of orthorhombic media and in VTI media means that the symmetry-plane attenuation coefficients of all three modes can be obtained by adapting the VTI equations of Zhu and Tsvankin (2004, 2005a). While the exact attenuation coefficients are rather complicated even for VTI models and do not provide insight into the influence of various attenuation-anisotropy parameters, much simpler solutions can be found under the following assumptions:

1. The magnitude of attenuation measured by the inverse Q_{ij} values or the parameters $\mathcal{A}_{P0}$ and $\mathcal{A}_{S0}$ is small.
2. Attenuation anisotropy is weak, which implies that the absolute values of all attenuation-anisotropy parameters introduced above are much smaller than unity.
3. Velocity anisotropy is also weak, so the absolute values of all Tsvankin's (1997, 2001) anisotropy parameters are much smaller than unity.

In limit of weak attenuation and small anisotropy parameters $\gamma^{(2)}$ and $\gamma_Q^{(2)}$ ($|\gamma^{(2)}| \ll 1$ and $|\gamma_Q^{(2)}| \ll 1$), the approximate SH-wave attenuation coefficient in the $[x_1, x_3]$-plane can be written as

$$\mathcal{A}_{SH}^{(2)} = \bar{\mathcal{A}}_{S0}\,(1+\gamma_Q^{(2)}\sin^2\theta), \tag{30}$$

where

$$\bar{\mathcal{A}}_{S0} = \frac{1}{2Q_{44}} = \mathcal{A}_{S0}\,\frac{1+\gamma_Q^{(1)}}{1+\gamma_Q^{(2)}} \tag{31}$$

is the normalized attenuation coefficient for the vertically propagating shear wave polarized in the x_2-direction. Equation (30) is obtained by replacing the parameter γ_Q in the VTI result of Zhu and Tsvankin (2004, 2005a) by $\gamma_Q^{(2)}$ and using the appropriate isotropic reference value $\bar{\mathcal{A}}_{S0}$. Similarly, the corresponding linearized coefficient in the $[x_2, x_3]$-plane has the form

$$\mathcal{A}_{SH}^{(1)} = \mathcal{A}_{S0}\,(1+\gamma_Q^{(1)}\sin^2\theta). \tag{32}$$

It should be emphasized that the term "SH-wave" refers to two different shear modes in the vertical symmetry planes (Tsvankin, 1997, 2001). For example, if $c_{44} > c_{55}$, then the fast shear wave S_1 represents an SH-wave in the $[x_1, x_3]$-plane where it is polarized in the x_2-direction. For wave propagation in the $[x_2, x_3]$-plane, however, the S_1-wave becomes an SV mode that has an in-plane polarization vector.

The difference between the attenuation coefficients of the vertically traveling split shear waves can be quantified by the *attenuation splitting parameter* $\gamma_Q^{(S)}$:

$$\gamma_Q^{(S)} \equiv \frac{\bar{\mathcal{A}}_{S0}-\mathcal{A}_{S0}}{\mathcal{A}_{S0}} = \frac{\gamma_Q^{(1)}-\gamma_Q^{(2)}}{1+\gamma_Q^{(2)}}. \tag{33}$$

The definition (33) is analogous to that of the widely used S-wave velocity splitting parameter $\gamma^{(S)}$ (Tsvankin, 1997, 2001). To keep the parameter $\gamma_Q^{(S)}$ positive, we assume that $\bar{\mathcal{A}}_{S0} > \mathcal{A}_{S0}$; otherwise, $\bar{\mathcal{A}}_{S0}$ and $\mathcal{A}_{S0}$ in equation (33) have to be switched. Although $\gamma_Q^{(S)}$ would be redundant as part of our notation for attenuative orthorhombic media, this parameter should play an important role in the attenuation analysis of reflection shear-wave data.

Substituting the attenuation-anisotropy parameters $\epsilon_Q^{(2)}$ and $\delta_Q^{(2)}$ into the VTI equations of Zhu and Tsvankin (2004, 2005a) yields the following approximate attenuation coefficients of the P- and SV-waves in the $[x_1, x_3]$-plane:

$$\mathcal{A}_P^{(2)} = \mathcal{A}_{P0}\left(1 + \delta_Q^{(2)} \sin^2\theta \cos^2\theta + \epsilon_Q^{(2)} \sin^4\theta\right), \tag{34}$$

$$\mathcal{A}_{SV}^{(2)} = \mathcal{A}_{S0}\left(1 + \sigma_Q^{(2)} \sin^2\theta \cos^2\theta\right), \tag{35}$$

where

$$\sigma_Q^{(2)} \equiv \frac{1}{g_Q^{(2)}}\left[2(1 - g_Q^{(2)})\,\sigma^{(2)} + \frac{\epsilon_Q^{(2)} - \delta_Q^{(2)}}{g^{(2)}}\right], \tag{36}$$

$g^{(2)} \equiv \dfrac{c_{55}}{c_{33}}$, $g_Q^{(2)} \equiv \dfrac{Q_{33}}{Q_{55}} = \dfrac{\mathcal{A}_{S0}}{\mathcal{A}_{P0}}$, and $\sigma^{(2)} \equiv \dfrac{\epsilon^{(2)} - \delta^{(2)}}{g^{(2)}}$. The approximate attenuation coefficients in equations (34) and (35) have exactly the same form as the corresponding linearized phase-velocity equations (Thomsen, 1986). However, as discussed by Zhu and Tsvankin (2004, 2005a), the dependence of the attenuation-anisotropy parameter $\delta_Q^{(2)}$ on the real parts of the stiffness coefficients reflects the coupling between the attenuation and velocity anisotropy. In contrast, the anisotropic phase-velocity function is practically independent of attenuation (see below).

The coefficients $\mathcal{A}_P^{(1)}$ and $\mathcal{A}_{SV}^{(1)}$ in the $[x_2, x_3]$-plane are obtained in the same way from the VTI results by using the attenuation-anisotropy parameters $\epsilon_Q^{(1)}$ and $\delta_Q^{(1)}$. For example,

$$\mathcal{A}_P^{(1)} = \mathcal{A}_{P0}\left(1 + \delta_Q^{(1)} \sin^2\theta \cos^2\theta + \epsilon_Q^{(1)} \sin^4\theta\right). \tag{37}$$

6 P-WAVE ATTENUATION OUTSIDE THE SYMMETRY PLANES

This section is devoted to the analysis of the P-wave attenuation coefficient for phase directions outside the symmetry planes. While the shear-wave attenuation coefficients can be studied numerically by solving the Christoffel equation, the area of validity of such plane-wave solutions in describing shear-wave amplitudes is significantly reduced because of the influence of point singularities (Crampin, 1991).

6.1 Influence of attenuation on phase velocity

As pointed out above, the attenuation coefficients depend not just on the quality-factor elements Q_{ij} but also on the velocity-anisotropy parameters. In contrast, the presence of attenuation has an almost negligible influence on the phase-velocity function. This result, discussed by Zhu and Tsvankin (2004, 2005a) for VTI media, remains valid for the symmetry planes of the orthorhombic model. Here, we demonstrate that attenuation-related distortions of phase velocity remain negligible outside the symmetry planes as well.

In the limit of weak attenuation ($\dfrac{1}{Q_{ij}} \ll 1$), the real part of the Christoffel equation (A-2) can be simplified by dropping terms quadratic in the inverse Q components. The resulting equation (A-3) is identical to the Christoffel equation for the reference nonattenuative medium, both within and outside the symmetry planes.

To evaluate the contribution of the higher-order attenuation terms, we compute the exact P-wave phase velocity for two orthorhombic models with strong attenuation. For the first model, the attenuation is isotropic with the uncommonly low quality factor $Q_{33} = Q_{55} = 10$ (Figure 1). Still, the maximum attenuation-related change in the phase velocity is limited to 0.5%, which is equal to $\dfrac{1}{2Q_{33}^2}$.

The second model has the same real part of the stiffness matrix, but this time accompanied by pronounced attenuation anisotropy (Figure 2). Although the deviation of the phase-velocity function from that in the reference nonattenuative medium increases away from the vertical, it remains insignificant (up to 1%) for the whole range of polar and azimuthal phase angles. Note, however, that this discussion does not take into account attenuation-related velocity dispersion.

Hence, seismic processing for orthorhombic media with orthorhombic attenuation can be divided into two steps. First, one can perform anisotropic velocity analysis and estimation of Tsvankin's parameters without taking attenuation into account (Grechka and Tsvankin, 1999; Grechka et al., 1999; Bakulin et al., 2000). Then the reconstructed

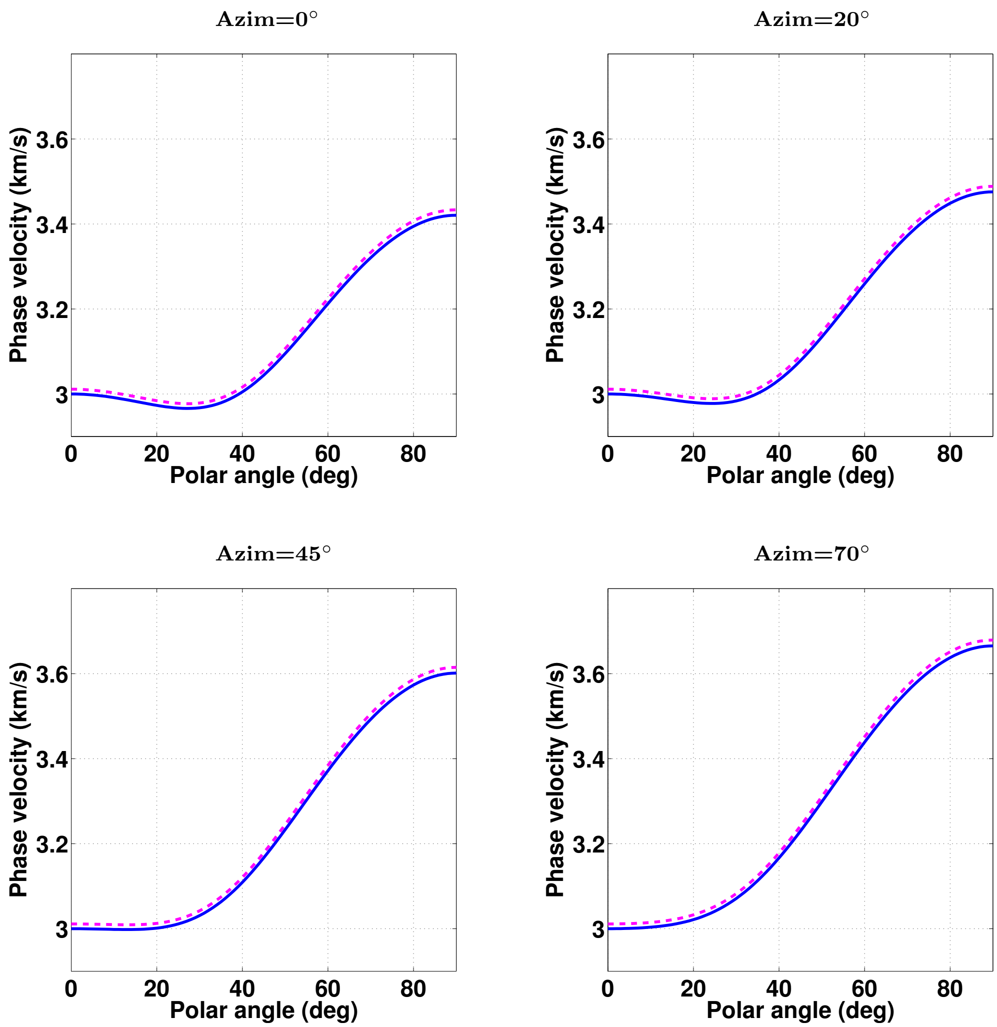

Figure 1. Influence of isotropic attenuation on the exact P-wave phase velocity computed from the Christoffel equation (5). Each plot corresponds to a fixed azimuthal phase angle. The solid curves mark the velocity for a nonattenuative orthorhombic model with the following Tsvankin's (1997, 2001) parameters: $V_{P0} = 3$ km/s, $V_{S0} = 1.5$ km/s, $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.15$, $\delta^{(1)} = 0.05$, $\delta^{(2)} = -0.1$, $\delta^{(3)} = 0.15$, $\gamma^{(1)} = 0.28$, and $\gamma^{(2)} = 0.15$. The dashed curves are computed for a model with the same velocity parameters and strong isotropic attenuation ($Q_{33} = Q_{55} = 10$; all attenuation-anisotropy parameters are set to zero).

anisotropic velocity model can be used in the processing of amplitude measurements and inversion for the attenuation-anisotropy parameters.

6.2 Approximate attenuation outside the symmetry planes

The linearized approximation for the P-wave attenuation coefficient can be extended for arbitrary propagation directions outside the symmetry plane under the assumption of weak attenuation and weak velocity and attenuation anisotropy. The approximate coefficient $\mathcal{A}_P$, expressed as a function of the polar phase angle θ and azimuthal phase angle ϕ, has the form (Appendix A)

$$\mathcal{A}_P(\theta,\phi) = \mathcal{A}_{P0}\left[1 + \delta_Q(\phi)\sin^2\theta\cos^2\theta + \epsilon_Q(\phi)\sin^4\theta\right], \tag{38}$$

where

$$\epsilon_Q(\phi) = \epsilon_Q^{(1)}\sin^4\phi + \epsilon_Q^{(2)}\cos^4\phi + (2\epsilon_Q^{(2)} + \delta_Q^{(3)})\sin^2\phi\cos^2\phi\,, \tag{39}$$

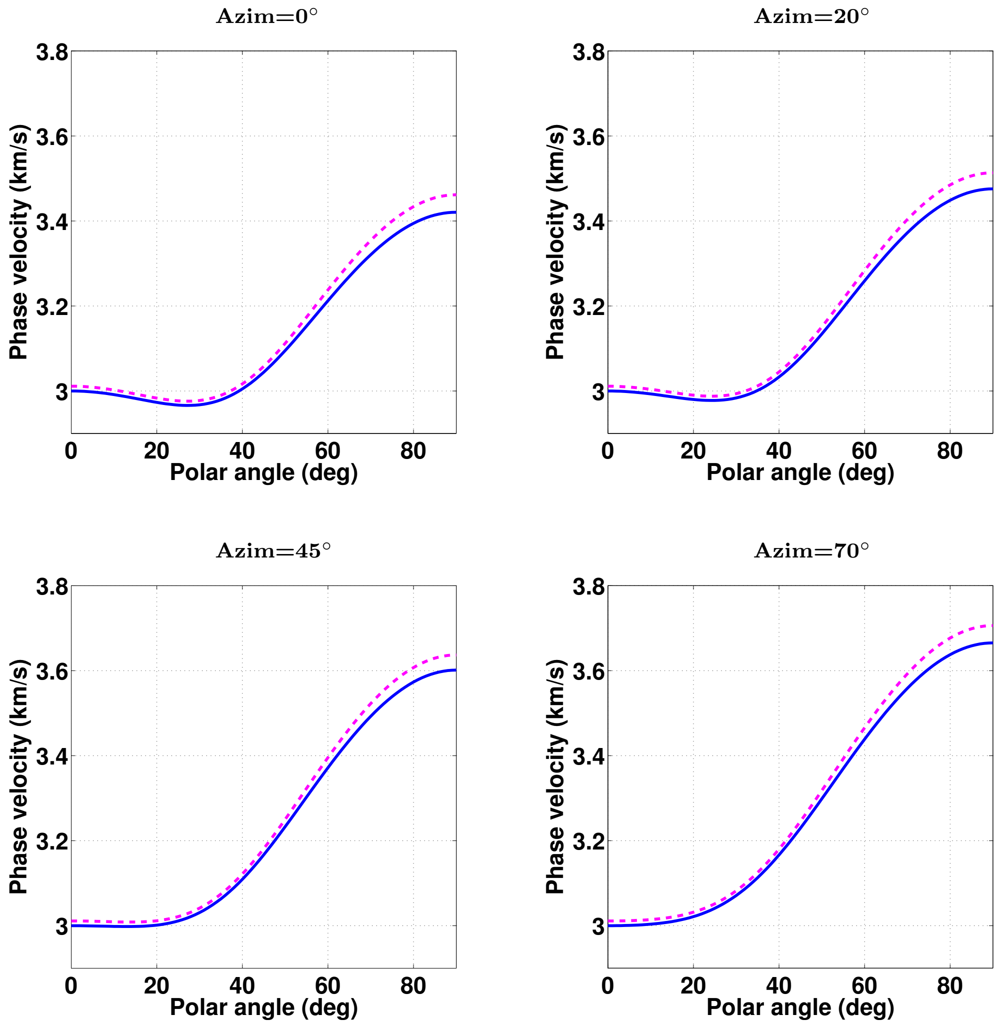

Figure 2. Influence of anisotropic attenuation on the exact P-wave phase velocity. The solid curves are the phase velocities for the nonattenuative orthorhombic model from Figure 1. The dashed curves are computed for a model with the same velocity parameters and strong orthorhombic attenuation: $Q_{33} = Q_{55} = 10$, $\epsilon_Q^{(1)} = \epsilon_Q^{(2)} = 0.8$, $\delta_Q^{(1)} = \delta_Q^{(2)} = \delta_Q^{(3)} = -0.5$, and $\gamma_Q^{(1)} = \gamma_Q^{(2)} = 0.8$.

$$\delta_Q(\phi) = \delta_Q^{(1)} \sin^2 \phi + \delta_Q^{(2)} \cos^2 \phi \,. \tag{40}$$

Evidently, the linearized P-wave attenuation coefficient in any vertical plane of orthorhombic media is described by the VTI equation (Zhu and Tsvankin, 2004, 2005a) with the azimuthally varying parameters $\epsilon_Q(\phi)$ and $\delta_Q(\phi)$. For wave propagation in the $[x_1, x_3]$-plane ($\phi = 0^\circ$), $\epsilon_Q = \epsilon_Q^{(2)}$, $\delta_Q = \delta_Q^{(2)}$, and equation (38) reduces to equation (34). Similarly, for the $[x_2, x_3]$-plane ($\phi = 90^\circ$), $\epsilon_Q = \epsilon_Q^{(1)}$, $\delta_Q = \delta_Q^{(1)}$, and equation (38) reduces to equation (37).

Remarkably, equations (38)–(40) have exactly the same form as the linearized P-wave phase-velocity equations (1.107)–(1.109) in Tsvankin (2001). This similarity is explained by the identical (orthorhombic) symmetry imposed on both the real and imaginary parts of the stiffness matrix and the assumption of homogeneous wave propagation. However, an important difference between the coefficient $\mathcal{A}_P$ and phase velocity is that the parameters $\delta_Q^{(1)}$, $\delta_Q^{(2)}$, and $\delta_Q^{(3)}$ include a contribution of the velocity anisotropy, while the velocity function is practically independent of attenuation. Also, as discussed below, the exact coefficient $\mathcal{A}_P$ is influenced by the velocity-anisotropy parameters even for fixed values of $\epsilon_Q^{(1,2)}$ and $\delta_Q^{(1,2,3)}$.

6.3 Attenuation for VTI and HTI media

Transversely isotropic models with both vertical (VTI) and horizontal (HTI) symmetry axis can be treated as special cases of orthorhombic media. For VTI media with VTI attenuation, all vertical planes are identical, and there is no velocity or attenuation variation in the horizontal (isotropy) plane:

$$\begin{aligned}
\epsilon^{(1)} &= \epsilon^{(2)} = \epsilon, \quad \epsilon_Q^{(1)} = \epsilon_Q^{(2)} = \epsilon_Q \\
\delta^{(1)} &= \delta^{(2)} = \delta, \quad \delta_Q^{(1)} = \delta_Q^{(2)} = \delta_Q, \\
\gamma^{(1)} &= \gamma^{(2)} = \gamma, \quad \gamma_Q^{(1)} = \gamma_Q^{(2)} = \gamma_Q, \\
\delta^{(3)} &= 0, \quad \delta_Q^{(3)} = 0\,.
\end{aligned}$$

Then equations (38)–(40) yield the VTI result (Zhu and Tsvankin, 2004, 2005a): $\epsilon_Q(\phi) = \epsilon_Q$, $\delta_Q(\phi) = \delta_Q$, and

$$\mathcal{A}_P^{\mathrm{VTI}} = \mathcal{A}_{P0}\left(1 + \delta_Q \sin^2\theta\cos^2\theta + \epsilon_Q \sin^4\theta\right). \tag{41}$$

Next, suppose that the symmetry axis of the TI medium (for both velocity and attenuation) points in the x_1-direction. In this case, the isotropy plane coincides with the $[x_2, x_3]$-plane, and

$$\begin{aligned}
\epsilon^{(1)} &= 0, \quad \epsilon_Q^{(1)} = 0, \\
\delta^{(1)} &= 0, \quad \delta_Q^{(1)} = 0, \\
\gamma^{(1)} &= 0, \quad \gamma_Q^{(1)} = 0\,.
\end{aligned}$$

For this HTI model, the parameters $\delta^{(3)}$ and $\delta_Q^{(3)}$ are not independent because the $[x_1, x_2]$-plane is equivalent to the $[x_1, x_3]$-plane. If the velocity anisotropy is weak, $\delta^{(3)} = \delta^{(2)} - 2\epsilon^{(2)}$ (Tsvankin, 1997, 2001). For weak attenuation anisotropy, $\delta_Q^{(3)} = \delta_Q^{(2)} - 2\epsilon_Q^{(2)}$, and equation (39) becomes $\epsilon_Q(\phi) = \epsilon_Q^{(2)}\cos^4\phi + \delta_Q^{(2)}\sin^2\phi\cos^2\phi$. Then the P-wave attenuation coefficient (38) takes the form

$$\mathcal{A}_P^{\mathrm{HTI}} = \mathcal{A}_{P0}\left[1 + \delta_Q^{(2)}\cos^2\phi\sin^2\theta\cos^2\theta + \left(\epsilon_Q^{(2)}\cos^4\phi + \delta_Q^{(2)}\sin^2\phi\cos^2\phi\right)\sin^4\theta\right]. \tag{42}$$

6.4 Parameters for P-wave attenuation

The linearized P-wave attenuation coefficient (38) does not contain the parameters $\mathcal{A}_{S0}$, $\gamma_Q^{(1)}$, and $\gamma_Q^{(2)}$, which are primarily responsible for shear-wave attenuation. An important practical issue is whether or not this conclusion remains valid for models with strong attenuation and pronounced velocity and attenuation anisotropy. As illustrated by Figure 3, the dependence of $\mathcal{A}_P$ on the shear-wave vertical attenuation coefficient $\mathcal{A}_{S0}$ becomes noticeable only for extremely high attenuation (i.e., uncommonly small values of Q_{55}). The influence of the parameters $\gamma_Q^{(1)}$ and $\gamma_Q^{(2)}$ on the coefficient $\mathcal{A}_P$ (not shown here) for typical moderately attenuative models is also negligible.

Therefore, for a fixed orientation of the symmetry planes and fixed velocity parameters, P-wave attenuation is controlled by the reference value $\mathcal{A}_{P0}$ and five attenuation-anisotropy parameters – $\epsilon_Q^{(1)}$, $\epsilon_Q^{(2)}$, $\delta_Q^{(1)}$, $\delta_Q^{(2)}$, and $\delta_Q^{(3)}$. An equivalent result was obtained for velocity anisotropy by Tsvankin (1997, 2001), who showed that the P-wave phase-velocity function in orthorhombic media is governed just by the vertical velocity and five ϵ and δ parameters. However, while the velocity function is almost independent of attenuation, the P-wave attenuation coefficient does depend on the velocity anisotropy, even if all relevant attenuation-anisotropy parameters are held constant (see below).

6.5 Accuracy of the linearized solution

To evaluate the accuracy of the weak-anisotropy approximation (38) outside the symmetry planes, we compare it with the exact coefficient $\mathcal{A}_P$ [equation (5)] for a model with pronounced orthorhombic attenuation (Figure 4). The velocity parameters correspond to the moderately anisotropic model of Schoenberg and Helbig (1997). Since no measurements of the attenuation-anisotropy parameters are available, each of them was set to be twice as large as the corresponding velocity-anisotropy parameter (e.g., $\epsilon_Q^{(2)} = 2\epsilon^{(2)}$).

As expected, the weak-anisotropy approximation gives satisfactory results for near-vertical propagation directions with polar angles up to about 30°. The error becomes more significant for intermediate propagation angles in the range $30° < \theta < 75°$. When the vertical incidence plane is close to either vertical symmetry plane (i.e., the azimuth ϕ approaches 0° or 90°), the approximate solution also yields an accurate estimate of $\mathcal{A}_P$ near the horizontal direction. Overall, the error of the weak-anisotropy approximation for the full range of polar and azimuthal angles is less than 10%. Note that while the velocity anisotropy for this model is moderate (both $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are about 0.3),

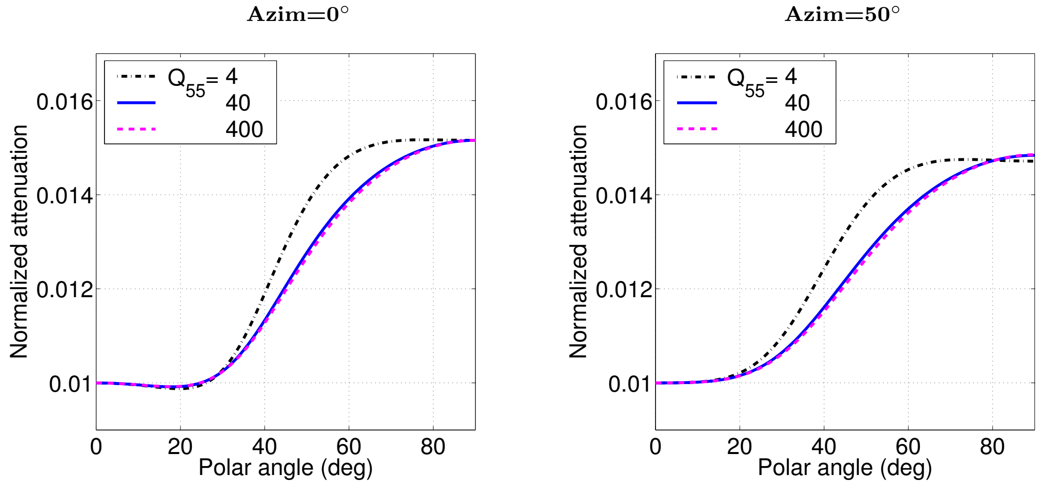

Figure 3. Influence of the parameter $\mathcal{A}_{S0} = 1/(2Q_{55})$ (marked on the plot) on the normalized P-wave attenuation coefficient. The velocity parameters correspond to an orthorhombic model formed by vertical cracks embedded in a VTI background (Schoenberg and Helbig, 1997): $V_{P0} = 2.437$ km/s, $V_{S0} = 1.265$ km/s, $\epsilon^{(1)} = 0.329$, $\epsilon^{(2)} = 0.258$, $\delta^{(1)} = 0.083$, $\delta^{(2)} = -0.078$, $\delta^{(3)} = -0.106$, $\gamma^{(1)} = 0.182$, and $\gamma^{(2)} = 0.0455$. The P-wave vertical attenuation coefficient is $\mathcal{A}_{P0} = 0.01$ ($Q_{33} = 50$); each attenuation-anisotropy parameter is twice the corresponding velocity-anisotropy parameter: $\epsilon_Q^{(1)} = 0.658$, $\epsilon_Q^{(2)} = 0.516$, $\delta_Q^{(1)} = 0.166$, $\delta_Q^{(2)} = -0.156$, $\delta_Q^{(3)} = -0.212$, $\gamma_Q^{(1)} = 0.364$, and $\gamma_Q^{(2)} = 0.091$.

the attenuation anisotropy is much more pronounced. This and other tests for a representative set of orthorhombic models confirm that equation (38) adequately describes P-wave attenuation under the assumption of homogeneous wave propagation.

To identify the source of errors in the weak-anisotropy approximation, we repeat the test in Figure 4 using a purely isotropic velocity model (Figure 5). The approximate solution (dashed lines) in Figure 5 coincides with that in Figure 4 because both models have identical attenuation-anisotropy parameters. The exact coefficient $\mathcal{A}_P$ (solid lines), however, is influenced by the velocity-anisotropy parameters in such a way that the error of the weak-anisotropy approximation becomes much smaller when the velocity field is isotropic (Figure 5).

Hence, the accuracy of the approximation (38) is controlled primarily by the strength of the velocity anisotropy, even if the magnitude of the attenuation anisotropy is much higher. This can be explained by the multiple linearizations in the velocity-anisotropy parameters involved in deriving equations (A4), (A7), and (A8).

It should be emphasized that the influence of different subsets of the velocity-anisotropy parameters on the attenuation coefficient varies with the azimuth ϕ. As illustrated in Figure 6, the contribution of the velocity-anisotropy parameters defined in the $[x_1, x_3]$-plane (the azimuth $\phi = 0°$) to the coefficient $\mathcal{A}_P$ decreases away from that plane and completely vanishes in the orthogonal direction. Note that according to the Christoffel equation (17), the P-wave attenuation coefficient in the $[x_2, x_3]$-plane ($\phi = 90°$) is indeed fully independent of the velocity- and attenuation-anisotropy parameters defined in the other two symmetry planes. Similarly, the maximum influence on $\mathcal{A}_P$ of the velocity-anisotropy parameters defined in the $[x_2, x_3]$-plane is observed for azimuths close to 90°.

7 DISCUSSION AND CONCLUSIONS

The attenuation coefficients of P-, S_1-, and S_2-waves in orthorhombic media with orthorhombic attenuation depend on the orientation of the symmetry planes, nine velocity parameters and nine components of the quality-factor matrix. The large number of independent parameters, compounded by the coupling between the attenuation and velocity anisotropy, makes the description of orthorhombic attenuation extremely difficult. Here, we demonstrated that the analysis of attenuation coefficients can be significantly simplified by introducing attenuation-anisotropy parameters similar to Tsvankin's parameters for the orthorhombic velocity function.

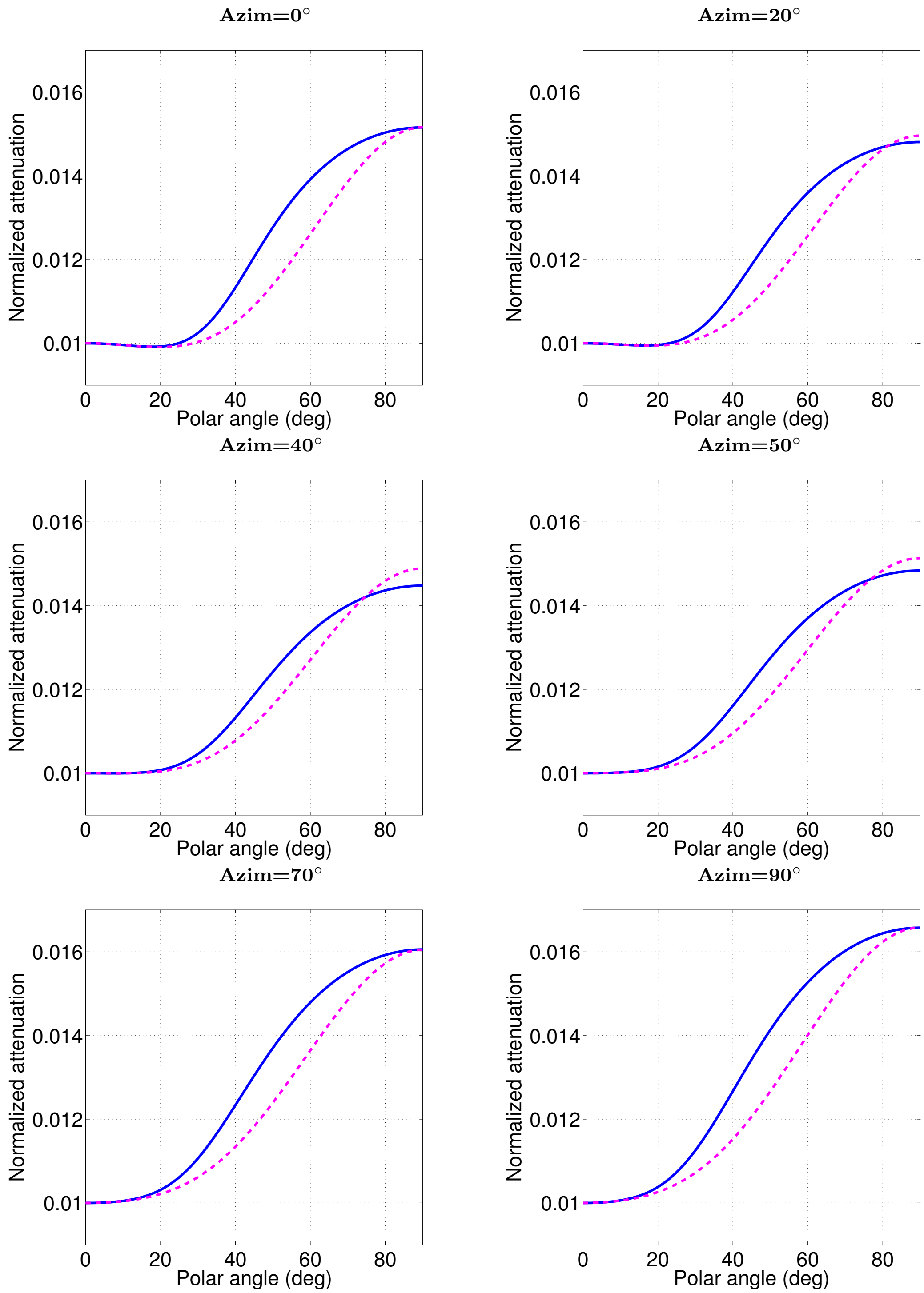

Figure 4. Comparison of the exact coefficient $\mathcal{A}_P$ (solid curves) with the linearized approximation (38) (dashed) for an orthorhombic medium with orthorhombic attenuation. The model parameters are the same as in Figure 3 ($Q_{55} = 40$).

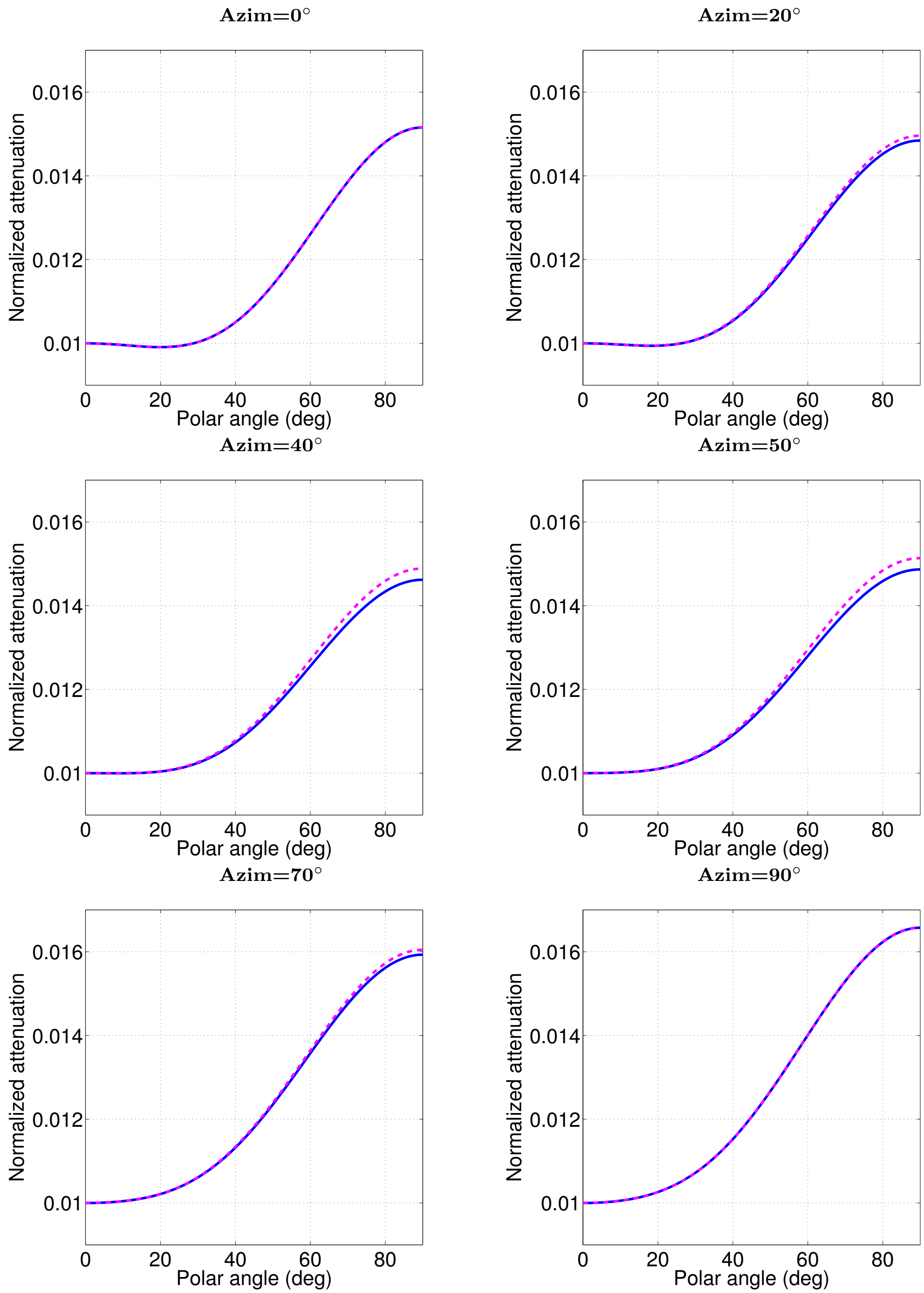

Figure 5. Comparison of the exact coefficient $\mathcal{A}_P$ (solid curves) with the linearized approximation (38) (dashed) for a medium with orthorhombic attenuation but a purely isotropic velocity function. The attenuation parameters are the same as in Figures 3 and 4, but the velocity $V_{P0} = 2.437$ km/s is constant in all directions.

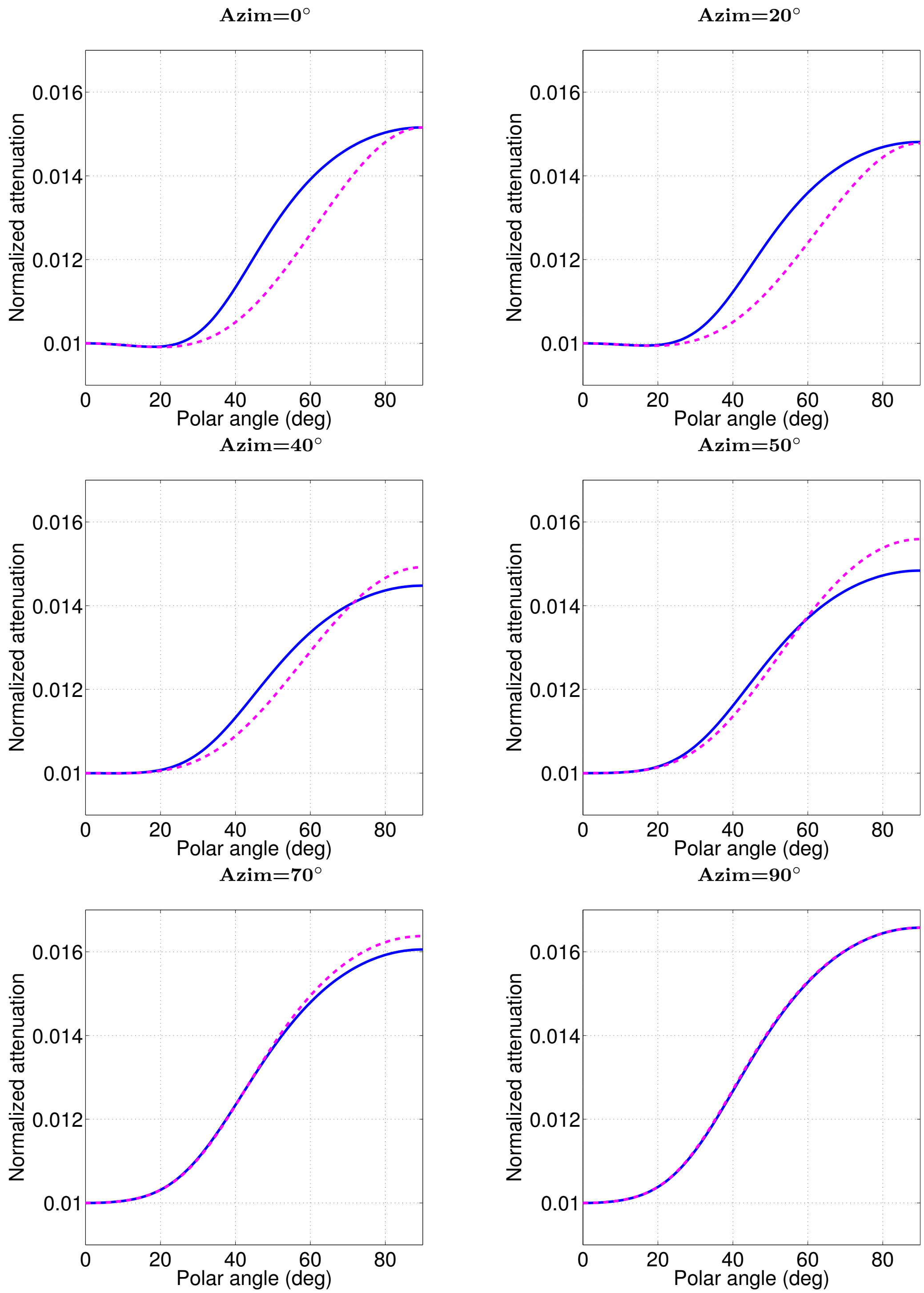

Figure 6. Influence of the velocity anisotropy on the exact attenuation coefficient $\mathcal{A}_P$. The solid curves are computed for the orthorhombic model with orthorhombic attenuation from Figures 3 and 4. The dashed curves are obtained by setting to zero the velocity-anisotropy parameters $\epsilon^{(2)}$, $\delta^{(2)}$, and $\gamma^{(2)}$ defined in the $[x_1, x_3]$-plane (azimuth $= 0°$); all other model parameters are unchanged.

The equivalence between the Christoffel equation in the symmetry planes of orthorhombic and VTI media, established previously for purely elastic media, holds in the presence of orthorhombic attenuation. Therefore, the symmetry-plane attenuation coefficients of all three modes can be obtained by simply adapting the known VTI equations. Also, the Thomsen-style VTI notation of Zhu and Tsvankin (2004, 2005a) can be extended to orthorhombic media following the approach suggested by Tsvankin (1997, 2001) for velocity anisotropy. The set of attenuation-anisotropy parameters introduced here includes two reference (isotropic) P- and S-wave attenuation coefficients in the x_3 direction, $\mathcal{A}_{P0}$ and $\mathcal{A}_{S0}$, and seven dimensionless anisotropy parameters, $\epsilon_Q^{(1,2)}$, $\delta_Q^{(1,2,3)}$, and $\gamma_Q^{(1,2)}$.

Adaptation of the linearized VTI equations leads to concise expressions for the symmetry-plane attenuation coefficients of P-, S_1-, and S_2-waves valid for weak attenuation and weak velocity and attenuation anisotropy. Furthermore, linearization of the Christoffel equation in the attenuation-anisotropy parameters yields the approximate P-wave attenuation coefficient $\mathcal{A}_P$ outside the symmetry planes as a simple function of $\mathcal{A}_{P0}$, $\epsilon_Q^{(1,2)}$, and $\delta_Q^{(1,2,3)}$. Interestingly, the linearized coefficient $\mathcal{A}_P$ expressed through the phase angles and attenuation-anisotropy parameters has the same form as the approximate P-wave phase-velocity function in terms of Tsvankin's velocity parameters. Also, as is the case for velocity anisotropy, the approximate P-wave attenuation coefficient in each vertical plane of orthorhombic media is given by the VTI equation with azimuthally varying parameters ϵ_Q and δ_Q.

This equivalence between the linearized equations for attenuation and velocity anisotropy stems from the identical (orthorhombic) symmetry of the real and imaginary parts of the stiffness tensor and the assumption of homogeneous wave propagation. Still, there are important differences between the treatment of velocity and attenuation anisotropy. Our analysis shows that in the absence of pronounced velocity dispersion the influence of attenuation (i.e., of the imaginary part of the stiffness tensor) on velocity is practically negligible. In contrast, the definitions of the attenuation-anisotropy parameters $\delta_Q^{(1,2,3)}$ include the velocity-anisotropy parameters $\delta^{(1,2,3)}$.

Also, although the velocity anisotropy does not explicitly contribute to the linearized expressions for attenuation, the exact attenuation coefficient $\mathcal{A}_P$ does vary with the velocity-anisotropy parameters even for fixed values of $\delta_Q^{(1,2,3)}$. Moreover, the accuracy of the linearized equation for $\mathcal{A}_P$ is controlled to a large degree by the strength of the velocity anisotropy. Numerical tests demonstrate that the approximate $\mathcal{A}_P$ remains close to the exact value even for strongly attenuative media provided the velocity anisotropy is relatively weak.

Thus, the P-wave attenuation coefficient is primarily governed by the orientation of the symmetry planes and six (instead of nine) attenuation-anisotropy parameters: $\mathcal{A}_{P0}$, $\epsilon_Q^{(1,2)}$, and $\delta_Q^{(1,2,3)}$. Still, due to the non-negligible influence of the velocity anisotropy on $\mathcal{A}_P$, inversion of attenuation measurements for orthorhombic media cannot be performed without anisotropic velocity analysis. Also, note that knowledge of the velocity field is required to correct for the difference between the phase attenuation coefficient studied here and the group attenuation coefficient responsible for the amplitude decay along seismic rays (Zhu and Tsvankin, 2004).

8 ACKNOWLEDGMENTS

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REFERENCES

Akbar, N., Dvorkin, J., and Nur, A., 1993, Relating P-wave attenuation to permeability: Geophysics, **58**, 20–29.

Bakulin, A., Grechka, V., and Tsvankin, I., 2000, Estimation of fracture parameters from reflection seismic data. Part II: Fractured models with orthorhombic symmetry: Geophysics, **65**, 1803–1817.

Carcione, J. M., 2001, Wave fields in real media: wave propagation in anisotropic, anelastic, and porous media: Pergamon Press.

Crampin, S., 1991, Effects of singularities on shear-wave propagation in sedimentary basins: Geophysical Journal International, **107**, 531–543.

Grechka, V., Theophanis, S., and Tsvankin, I., 1999, Joint inversion of P– and PS–waves in orthorhombic media: Theory and a physical-modeling study: Geophysics, **64**, 146–161.

Grechka, V., and Tsvankin, I., 1999, 3-D moveout velocity analysis and parameter estimation for orthorhombic media: Geophysics, **64**, 820–837.

Lynn, H. B., Campagna, D., Simon, K. M., and Beckham, W. E., 1999, Relationship of P-wave seismic attributes, azimuthal anisotropy, and commercial gas pay in 3-D P-wave multiazimuth data, Rulison Field, Piceance Basin, Colorado: Geophysics, **64**, 1293–1311.

Mavko, G. M., and Nur, A., 1979, Wave attenuation in partially saturated rocks: Geophysics, **44**, 161–178.

Pointer T., Liu, E., and Crampin, S., 1996, Seismic wave propagation in media with interconnected cracks and pores: 66nd Annual International Meeting, Society of Exploration Geophysicists, Expanded Abstracts, 1846–1849.

Schoenberg, M., and Helbig, K., 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954–1974.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966.
Tsvankin, I., 1997, Anisotropic parameters and P-wave velocity for orthorhombic media: Geophysics, **62**, 1292–1309.
Tsvankin, I., 2001, Seismic signatures and analysis of reflection data in anisotropic media: Elsevier.
Zhu, Y., and Tsvankin, I., 2004, Plane-wave propagation and radiation patterns in attenuative TI media: 74th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 139–142.
Zhu, Y., and Tsvankin, I., 2005a, Plane-wave propagation in attenuative TI media: Geophysics, under review.
Zhu, Y., and Tsvankin, I., 2005b, Physical modeling and analysis of P-wave attenuation anisotropy in transversely isotropic media: CWP Project Review (this volume).

APPENDIX A: APPROXIMATE ATTENUATION OUTSIDE THE SYMMETRY PLANES OF ORTHORHOMBIC MEDIA

The complex Christoffel equation (5) for homogeneous wave propagation outside the symmetry planes can be rewritten as

$$\begin{aligned}
&[\,(c_{11}n_1^2+c_{66}n_2^2+c_{55}n_3^2)\mathcal{K}_{1,(1,6,5)}-\rho V^2+i(c_{11}n_1^2+c_{66}n_2^2+c_{55}n_3^2)\mathcal{K}_{2,(1,6,5)}]\\
&\cdot\{\,[\,(c_{66}n_1^2+c_{22}n_2^2+c_{44}n_3^2)\mathcal{K}_{1,(6,2,4)}-\rho V^2+i(c_{66}n_1^2+c_{22}n_2^2+c_{44}n_3^2)\mathcal{K}_{2,(6,2,4)}]\\
&\cdot\,[\,(c_{55}n_1^2+c_{44}n_2^2+c_{33}n_3^2)\mathcal{K}_{1,(5,4,3)}-\rho V^2+i(c_{55}n_1^2+c_{44}n_2^2+c_{33}n_3^2)\mathcal{K}_{2,(5,4,3)}]-\\
&\quad[\,(c_{23}+c_{44})n_2n_3(\mathcal{K}_{1,(23,44)}+i\mathcal{K}_{2,(23,44)})]^2\}\\
&-[\,(c_{12}+c_{66})n_1n_2(\mathcal{K}_{1,(12,66)}+i\mathcal{K}_{2,(12,66)})]\\
&\cdot\{\,[\,(c_{12}+c_{66})n_1n_2(\mathcal{K}_{1,(12,66)}+i\mathcal{K}_{2,(12,66)})]\\
&\cdot\,[\,(c_{55}n_1^2+c_{44}n_2^2+c_{33}n_3^2)\mathcal{K}_{1,(5,4,3)}-\rho V^2+i(c_{55}n_1^2+c_{44}n_2^2+c_{33}n_3^2)\mathcal{K}_{2,(5,4,3)}]-\\
&\quad[\,(c_{13}+c_{55})n_1n_3(\mathcal{K}_{1,(13,55)}+i\mathcal{K}_{2,(13,55)})]\cdot[(c_{23}+c_{44})n_2n_3(\mathcal{K}_{1,(23,44)}+i\mathcal{K}_{2,(23,44)})]\}\\
&+[\,(c_{13}+c_{55})n_1n_3(\mathcal{K}_{1,(13,55)}+i\mathcal{K}_{2,(13,55)})]\\
&\cdot\{\,[\,(c_{12}+c_{66})n_1n_2(\mathcal{K}_{1,(12,66)}+i\mathcal{K}_{2,(12,66)})]\cdot[(c_{23}+c_{44})n_2n_3(\mathcal{K}_{1,(23,44)}+i\mathcal{K}_{2,(23,44)})]-\\
&\quad[\,(c_{13}+c_{55})n_1n_3(\mathcal{K}_{1,(13,55)}+i\mathcal{K}_{2,(13,55)})]\\
&\quad[\,(c_{66}n_1^2+c_{22}n_2^2+c_{44}n_3^2)\mathcal{K}_{1,(6,2,4)}-\rho V^2+i(c_{66}n_1^2+c_{22}n_2^2+c_{44}n_3^2)\mathcal{K}_{2,(6,2,4)}]\}=0\,,
\end{aligned} \tag{A1}$$

where

$$\begin{aligned}
\mathcal{K}_1 &= 1-\mathcal{A}^2+\frac{2}{Q_{33}}\mathcal{A}\,, \qquad \mathcal{K}_2=\frac{1-\mathcal{A}^2}{Q_{33}}-2\mathcal{A}\,,\\
\mathcal{K}_{1,(i,j,l)} &= \mathcal{K}_1+2\frac{\Delta_{(i,j,l)}}{Q_{33}}\mathcal{A}\,, \quad \mathcal{K}_{2,(i,j,l)}=\mathcal{K}_2+\frac{\Delta_{(i,j,l)}}{Q_{33}}(1-\mathcal{A}^2)\,,\\
\mathcal{K}_{1,(ij,kl)} &= \mathcal{K}_1+2\frac{\Delta_{(ij,kl)}}{Q_{33}}\mathcal{A}\,, \quad \mathcal{K}_{2,(ij,kl)}=\mathcal{K}_2+\frac{\Delta_{(ij,kl)}}{Q_{33}}(1-\mathcal{A}^2)\,,\\
\Delta_{(i,j,l)} &= \frac{c_{ii}n_1^2\dfrac{Q_{33}-Q_{ii}}{Q_{ii}}+c_{jj}n_2^2\dfrac{Q_{33}-Q_{jj}}{Q_{jj}}+c_{ll}n_3^2\dfrac{Q_{33}-Q_{ll}}{Q_{ll}}}{c_{ii}n_1^2+c_{jj}n_2^2+c_{ll}n_3^2}\,,\\
\Delta_{(ij,kl)} &= \frac{c_{ij}\dfrac{Q_{33}-Q_{ij}}{Q_{ij}}+c_{kl}\dfrac{Q_{33}-Q_{kl}}{Q_{kl}}}{c_{ij}+c_{kl}}\,.
\end{aligned}$$

Note that $\mathcal{A}\equiv k^I/k$ is on the order of the inverse Q-factor ($1/Q$). When the attenuation is weak ($\mathcal{A}\ll 1$), we obtain $\mathcal{K}_1\approx 1$ and $\mathcal{K}_2\approx \dfrac{1}{Q_{33}}-2\mathcal{A}$ by dropping the quadratic and higher-order terms in $\mathcal{A}$. Assuming that Q_{33} and Q_{55} are of the same order (the common case), weak attenuation anisotropy implies the same order for all components Q_{ij}.

Hence, the magnitude of the terms $\Delta_{(i,j,l)}$ and $\Delta_{(ij,kl)}$ cannot be much larger than unity. Then the terms $\frac{\Delta_{(i,j,l)}}{Q_{33}}\mathcal{A}$, $\frac{\Delta_{(ij,kl)}}{Q_{33}}\mathcal{A}$, $\frac{\Delta_{(i,j,l)}}{Q_{33}}\mathcal{A}^2$, and $\frac{\Delta_{(ij,kl)}}{Q_{33}}\mathcal{A}^2$ are either quadratic or cubic in $\mathcal{A}$. Dropping these terms yields $\mathcal{K}_{1,(i,j,l)} \approx 1$, $\mathcal{K}_{2,(i,j,l)} \approx \frac{1+\Delta_{(i,j,l)}}{Q_{33}} - 2\mathcal{A}$, $\mathcal{K}_{1,(ij,kl)} \approx 1$, and $\mathcal{K}_{2,(ij,kl)} \approx \frac{1+\Delta_{(ij,kl)}}{Q_{33}} - 2\mathcal{A}$.

Next, we denote $\mathcal{C}_{(i,j,l)} = c_{ii}n_1^2 + c_{jj}n_2^2 + c_{ll}n_3^2$ and $\mathcal{C}_{(ij,kl)} = (c_{ij} + c_{kl})n_i n_j$ and simplify equation (A1) for weak attenuation and weak attenuation anisotropy as

$$\begin{aligned}
&\left[\mathcal{C}_{(1,6,5)} - \rho V^2 + i\mathcal{C}_{(1,6,5)}\left(\frac{1+\Delta_{(1,6,5)}}{Q_{33}} - 2\mathcal{A}\right)\right] \\
&\left\{\left[\mathcal{C}_{(6,2,4)} - \rho V^2 + i\mathcal{C}_{(6,2,4)}(\frac{1+\Delta_{(6,2,4)}}{Q_{33}} - 2\mathcal{A})\right]\right. \\
&\left.\cdot\left[\mathcal{C}_{(5,4,3)} - \rho V^2 + i\mathcal{C}_{(5,4,3)}\left(\frac{1+\Delta_{(5,4,3)}}{Q_{33}} - 2\mathcal{A}\right)\right] - \mathcal{C}^2_{(23,44)}\left[1 + i\left(\frac{1+\Delta_{(23,44)}}{Q_{33}} - 2\mathcal{A}\right)\right]^2\right\} \\
&-\mathcal{C}_{(12,66)}\left[1 + \left(\frac{1+\Delta_{(12,66)}}{Q_{33}} - 2\mathcal{A}\right)\right] \\
&\left\{\mathcal{C}_{(12,66)}\left[1 + \left(\frac{1+\Delta_{(12,66)}}{Q_{33}} - 2\mathcal{A}\right)\right]\cdot\left[\mathcal{C}_{(5,4,3)} - \rho V^2 + i\mathcal{C}_{(5,4,3)}\left(\frac{1+\Delta_{(5,4,3)}}{Q_{33}} - 2\mathcal{A}\right)\right] -\right. \\
&\left.\mathcal{C}_{(13,55)}\left[1 + \left(\frac{1+\Delta_{(13,55)}}{Q_{33}} - 2\mathcal{A}\right)\right]\cdot\mathcal{C}_{(23,44)}\left[1 + i\left(\frac{1+\Delta_{(23,44)}}{Q_{33}} - 2\mathcal{A}\right)\right]\right\} \\
&+\mathcal{C}_{(13,55)}\left[1 + i\left(\frac{1+\Delta_{(13,55)}}{Q_{33}} - 2\mathcal{A}\right)\right] \\
&\left\{\mathcal{C}_{(12,66)}\left[1 + i\left(\frac{1+\Delta_{(12,66)}}{Q_{33}} - 2\mathcal{A}\right)\right]\cdot\mathcal{C}_{(23,44)}\left[1 + i\left(\frac{1+\Delta_{(23,44)}}{Q_{33}} - 2\mathcal{A}\right)\right] -\right. \\
&\left.\mathcal{C}_{(13,55)}\left[1 + i\left(\frac{1+\Delta_{(13,55)}}{Q_{33}} - 2\mathcal{A}\right)\right]\cdot\left[\mathcal{C}_{(6,2,4)} - \rho V^2 + i\mathcal{C}_{(6,2,4)}(\frac{1+\Delta_{(6,2,4)}}{Q_{33}} - 2\mathcal{A})\right]\right\} \\
&= 0\,.
\end{aligned} \tag{A2}$$

The real part of equation (A-2) is

$$\begin{aligned}
&(c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 - \rho V^2) \\
\cdot\;&\left[(c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 - \rho V^2)(c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2 - \rho V^2) - (c_{23} + c_{44})^2 n_2^2 n_3^2\right] \\
&-(c_{12} + c_{66})n_1 n_2 \\
\cdot\;&\left[(c_{12} + c_{66})n_1 n_2(c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2 - \rho V^2) - (c_{13} + c_{55})(c_{23} + c_{44})n_1 n_2 n_3^2\right] \\
&+(c_{13} + c_{55})n_1 n_3 \\
\cdot\;&\left[(c_{12} + c_{66})(c_{23} + c_{44})n_1 n_2 n_3^2 - (c_{13} + c_{55})n_1 n_3(c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 - \rho V^2)\right] = 0\,,
\end{aligned} \tag{A3}$$

which is identical to the Christoffel equation for the reference nonattenuative medium.

The normalized attenuation coefficient $\mathcal{A}$ is obtained from the imaginary part of equation (A2):

$$\mathcal{A} = \frac{1}{2Q_{33}}\left(1 + \frac{\mathcal{H}_u}{\mathcal{H}_d}\right), \tag{A4}$$

where

$$\begin{aligned}
\mathcal{H}_u = \;&\Delta_{(1,6,5)}\mathcal{C}_{(1,6,5)}\left[(\mathcal{C}_{(6,2,4)} - \rho V^2)(\mathcal{C}_{(5,4,3)} - \rho V^2) - \mathcal{C}^2_{(23,44)}\right] \\
&+\Delta_{(6,2,4)}\mathcal{C}_{(6,2,4)}\left[(\mathcal{C}_{(1,6,5)} - \rho V^2)(\mathcal{C}_{(5,4,3)} - \rho V^2) - \mathcal{C}^2_{(13,55)}\right] \\
&+\Delta_{(5,4,3)}\mathcal{C}_{(5,4,3)}\left[(\mathcal{C}_{(1,6,5)} - \rho V^2)(\mathcal{C}_{(6,2,4)} - \rho V^2) - \mathcal{C}^2_{(12,66)}\right] \\
&-2\Delta_{(13,55)}\mathcal{C}^2_{(13,55)}(\mathcal{C}_{(6,2,4)} - \rho V^2) - 2\Delta_{(12,66)}\mathcal{C}^2_{(12,66)}(\mathcal{C}_{(5,4,3)} - \rho V^2) \\
&-2\Delta_{(23,44)}\mathcal{C}^2_{(23,44)}(\mathcal{C}_{(1,6,5)} - \rho V^2)
\end{aligned}$$

$$+2\left(\Delta_{(13,55)} + \Delta_{(12,66)} + \Delta_{(23,44)}\right)\mathcal{C}_{(13,55)}\mathcal{C}_{(12,66)}\mathcal{C}_{(23,44)}\,, \tag{A5}$$

and

$$\mathcal{H}_d = \rho V^2 \;\Big[(\mathcal{C}_{(1,6,5)} - \rho V^2)(\mathcal{C}_{(6,2,4)} - \rho V^2) + (\mathcal{C}_{(1,6,5)} - \rho V^2)(\mathcal{C}_{(5,4,3)} - \rho V^2) + (\mathcal{C}_{(6,2,4)} - \rho V^2)(\mathcal{C}_{(5,4,3)} - \rho V^2) - \mathcal{C}^2_{(12,66)} - \mathcal{C}^2_{(13,55)} - \mathcal{C}^2_{(23,44)}\Big]\,. \tag{A6}$$

The term $\dfrac{\mathcal{H}_u}{\mathcal{H}_d}$ in equation (A4) can be expressed through the velocity- and attenuation-anisotropy parameters. Assuming that the anisotropy is weak for both the velocity and attenuation, we keep only the linear terms in all anisotropy parameters to obtain

$$\mathcal{H}_u = c_{33}(c_{33} - c_{55})^2\left[\epsilon^{(2)}_Q n_1^4 + \epsilon^{(1)}_Q n_2^4 + (2\epsilon^{(2)}_Q + \delta^{(3)}_Q)n_1^2 n_2^2 + \delta^{(2)}_Q n_1^2 n_3^2 + \delta^{(1)}_Q n_2^2 n_3^2\right]\,, \tag{A7}$$

$$\begin{aligned}\mathcal{H}_d \;=\; & c_{33}(c_{33} - c_{55}) \\ & \cdot\;\Big\{(c_{33} - c_{55})\left(1 + 2\epsilon^{(2)} n_1^4 + 2\epsilon^{(1)} n_2^4 + 2\delta^{(2)} n_1^2 n_3^2 + 2\delta^{(1)} n_2^2 n_3^2 + 4\epsilon^{(2)} n_1^2 n_2^2 + 2\delta^{(3)} n_1^2 n_2^2\right) \\ & +\; c_{33}\Big[\epsilon^{(1)}(-2n_2^2 + 6n_2^4) + \epsilon^{(2)}(-2n_1^2 + 6n_1^4 + 12n_1^2 n_2^2) \\ & \qquad +6\delta^{(1)} n_2^2 n_3^2 + 6\delta^{(2)} n_1^2 n_3^2 + 6\delta^{(3)} n_1^2 n_2^2\Big] \\ & +\; c_{55}\left[\gamma^{(1)}(-2 - 2n_2^2) + \gamma^{(2)}(2 - 2n_1^2)\right]\Big\}\,. \end{aligned} \tag{A8}$$

Note that since the term $\mathcal{H}_u$ is linear in the anisotropy parameters, it is sufficient to keep just the isotropic part of the term $\mathcal{H}_d$. Substitution of equations (A-7) and (A-8) into equation (A-4) yields the final form of the approximate P-wave attenuation coefficient given in the main text [equation (38)].